(U, N)-implications Satisfying U-Modus Ponens

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Outline

Introduction

- Fuzzy implication functions are used in fuzzy control and approximate reasoning to model fuzzy conditionals as well as to make inferences.

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- If we consider Zadeh's compositional rule of inference, Modus Ponens becomes essential in the process of managing forward inferences.

- Modus Ponens translated to the framework of fuzzy logic derives into:

$$T(x, I(x, y)) \leq y \quad \text{for all} \quad x, y \in [0, 1], \tag{MP}$$

  where T is a continuous t-norm and I is a fuzzy implication function.

Introduction

Modus Ponens in fuzzy logic: State of the art

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- Main studies are related to implications derived from t-norms and t-conorms: residual, (S, N)-implications, QL and D-implications.
- Many other kinds of implication functions can be considered, in particular implications derived from general aggregation functions.
- Recently, Modus Ponens has been studied for two kinds of implications derived from uninorms: RU-implications and (U, N)-implications.

- Uninorms were introduced in the framework of aggregation functions, but they have been also investigated as logical operators.

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- Uninorms have been extensively studied as generalizations of both t-norms and t-conorms.
- They have proved to be useful in fuzzy expert systems and also in fuzzy logic in general.
- Conjunctive uninorms are considered as conjunctions and the substitution of the t-norm by a uninorm in (MP) becomes natural.

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- The implication function in U-Modus Ponens must satisfy some properties fulfilled by RU-implications and (U, N)-implications.
- U-Modus Ponens has been investigated for RU-implications.

Goal: Deal with the same property for the case of (U, N)-implications.

Preliminaries

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Fuzzy implication functions

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(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Definition

A *uninorm* is a two-place function $U : [0,1]^2 \to [0,1]$ associative, commutative, increasing in each place and there exists some element $e \in [0,1]$, called *neutral element*, such that $U(e,x) = x$ for all $x \in [0,1]$.

If $e = 0$, U is a t-conorm. If $e = 1$, U is a t-norm.

If $e \in]0,1[$, U has the following structure:

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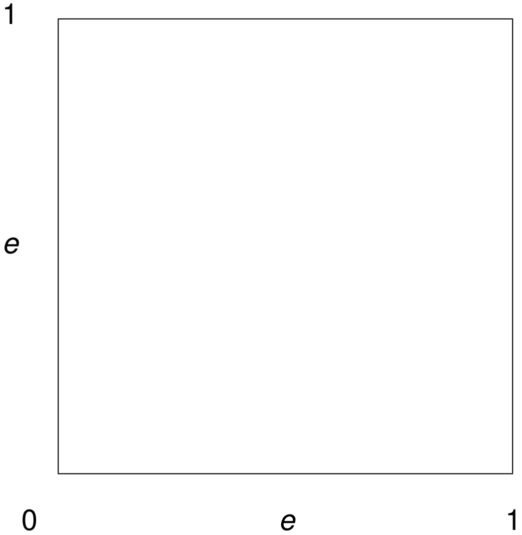
General structure of a uninorm



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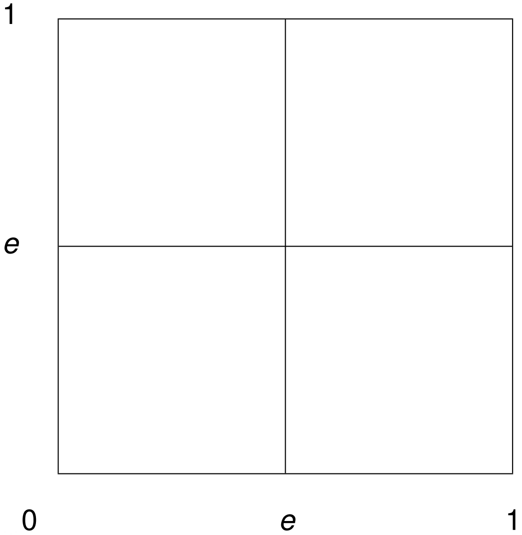

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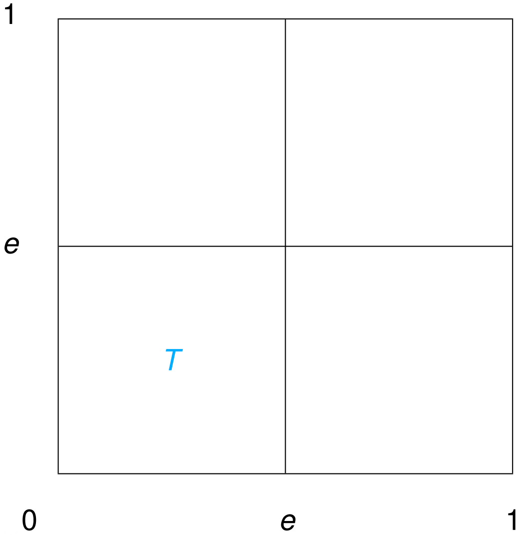

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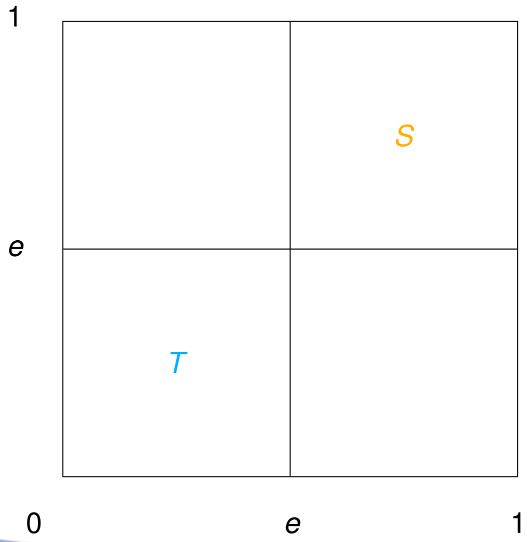

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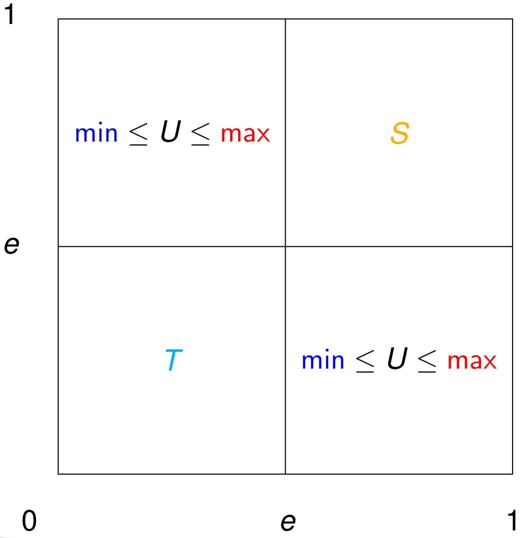

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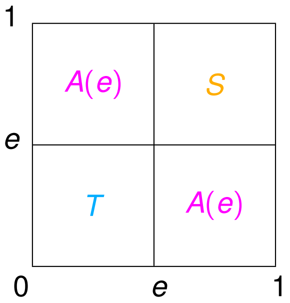
Classes of uninorms



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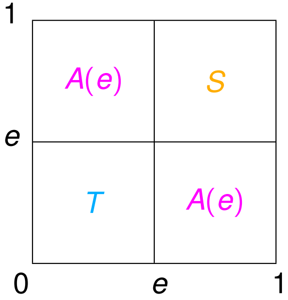

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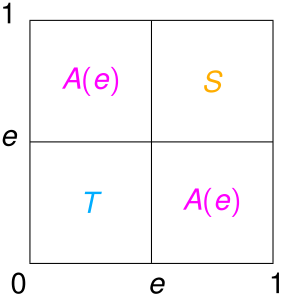

- $\mathcal{U}_{\text{lin}}$, if $U(x, y) \in \{x, y\}$ in $A(e)$

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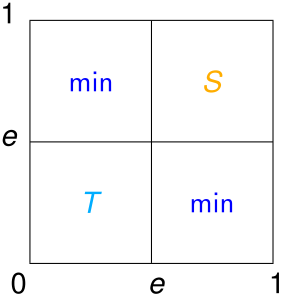

- $\mathcal{U}_{\text{lin}}$, if $U(x, y) \in \{x, y\}$ in $A(e)$
  - $\mathcal{U}_{\min}$, where $U(x, y) = \min(x, y)$ in $A(e)$
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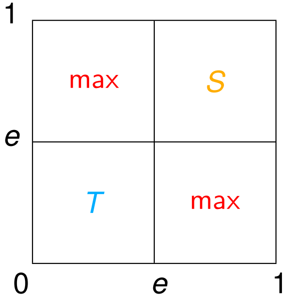

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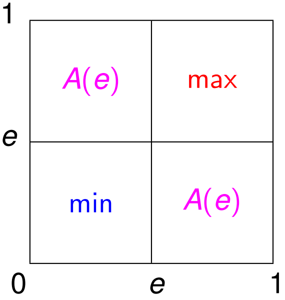

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  - $\mathcal{U}_{\text{ide}}$, satisfying $U(x, x) = x$

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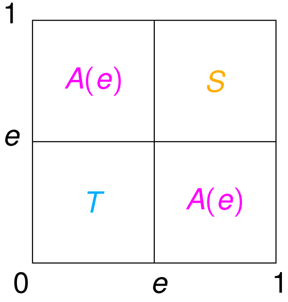

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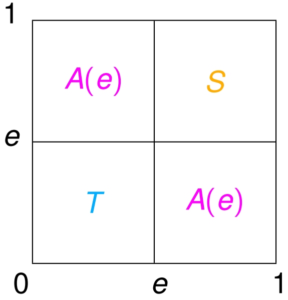

- $\mathcal{U}_{\text{lin}}$, if $U(x,y) \in \{x,y\}$ in $A(e)$
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  - $\mathcal{U}_{\cos}$, continuous in $]0,1[^2$

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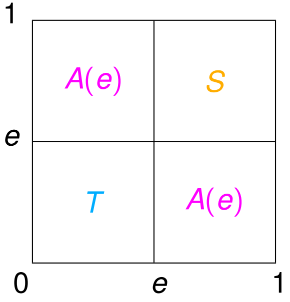
Classes of uninorms



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- $\mathcal{U}_{\text{cts}}$, with T i S continuous
  - $\mathcal{U}_{\cos}$, continuous in $]0, 1[^2$
    - ★ $\mathcal{U}_{\text{rep}}$, representable
      $U(x, y) = h^{-1}(h(x) + h(y))$

Preliminaries

Idempotent uninorms

Theorem

U is an idempotent uninorm with neutral element $e \in [0,1]$ if and only if there exists a non increasing function $g : [0,1] \to [0,1]$, symmetric with respect to the identity function, with $g(e) = e$, such that

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g^2(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g^2(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x), \end{cases}$$

being commutative in the points (x,y) such that $y = g(x)$ with $x = g^2(x)$.

Notation: $U \equiv \langle g, e \rangle_{\text{ide}}$.

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Representable uninorms

Definition

A uninorm U, with neutral element $e \in \,]0,1[\,$, is called *representable* if there exists a strictly increasing function $h : [0,1] \to [-\infty, +\infty]$ (called an *additive generator* of U, which is unique up to a multiplicative constant $k > 0$), with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$, such that U is given by

$$U(x, y) = h^{-1}(h(x) + h(y))$$

for all $(x, y) \in [0,1]^2 \setminus \{(0,1), (1,0)\}$. We have either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$.

Notation: $U \equiv \langle e, h \rangle_{\text{rep}}$.

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Structure of Uninorms continuous in $]0, 1[^2$

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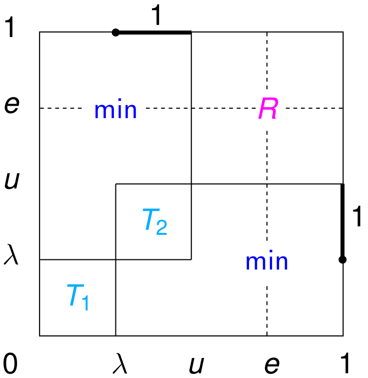

$\mathcal{U}_{\mathrm{cos,min}}$

$$U \equiv \langle T_1, \lambda, T_2, u, (R, e)\rangle_{\mathrm{cos,min}}$$

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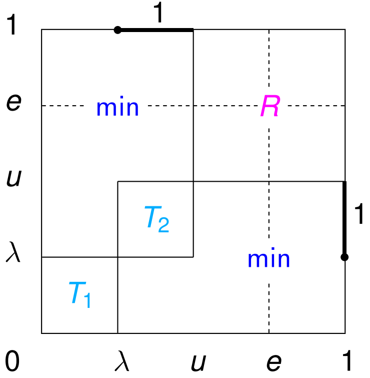

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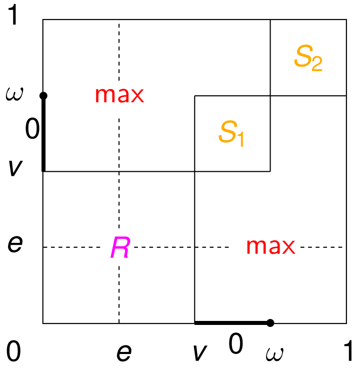

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U-Modus Ponens

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Definition

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I an implication function, U a uninorm.

I satisfies the *Modus Ponens property* with respect to U if

$$U(x, I(x, y)) \leq y \quad \text{for all } x, y \in [0, 1]. \tag{1}$$

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Remark

I an implication function, U a uninorm. If I is a *U-conditional* then U must be conjunctive.
If $I_{U',N}$ is a fuzzy implication, U' must be disjunctive.

U-Modus Ponens for (U, N)-implications

Proposition

$U \equiv \langle T, e, S\rangle$ a conjunctive uninorm, $U' \equiv \langle T', e', S'\rangle$ a disjunctive one, N a fuzzy negation and $I_{U',N}$ the corresponding (U, N)-implication. If $I_{U',N}$ satisfies the U-Modus ponens respect to U, the following items hold:

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2. $U'(N(x), y) \leq e$ *for all* x, y *such that* $e \leq y < x$*. In particular, it must be* $U'(0, y) < e$ *for all* $e \leq y < 1$.

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3. $U(x, N(x)) \leq e'$ *for all* $x \in [0, 1]$*. If N has a fixed point* e_N *then it must be* $U(e_N, e_N) \leq e'$.

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4. N *must be non-filling,* $N(x) < 1$ *for all* $x > 0$.

Proposition

$U \equiv \langle T, e, S \rangle$ conjunctive, $U' \equiv \langle T', e', S' \rangle$ disjunctive and locally internal on the boundary. N a fuzzy negation and $I_{U',N}$ its (U, N)-implication.
If $I_{U',N}$ satisfies (1) *with respect to U then:*

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If $I_{U',N}$ satisfies (1) with respect to U then:

- *It must be $U'(0, y) = 0$ for all $y < 1$.*
- *The natural negation of $I_{U',N}$ must be drastic fuzzy negation N_D given by $N_D(x) = 0$ for all $x < 1$.*

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- *U' can not be in $\mathcal{U}_{\max}$.*
- *If U' is in $\mathcal{U}_{cos,\max}$, say $U' \equiv \langle (R, e), v, S_1, \omega, S_2\rangle_{cos,\max}$, then it must be $\omega = 1$.*

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- *If U' is in $\mathcal{U}_{cos,\max}$, say $U' \equiv \langle (R, e), v, S_1, \omega, S_2 \rangle_{cos,\max}$, then it must be $\omega = 1$.*
- *If U' is idempotent, say $U' \equiv \langle g', e' \rangle_{ide}$, then it must be $g'(0) = 1$.*

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- idempotent uninorms with $g'(0) = 1$, or
- uninorms in $\mathcal{U}_{\text{cos,max}}$ with $\omega = 1$, or
- uninorms in $\mathcal{U}_{\text{cos,min}}$ with $\lambda = 0$ ($\lambda = 0$ to be U' disjunctive).

(U, N)-implication $I_{U',N}$ depends also on N:

Proposition

U' be a disjunctive uninorm in one of the classes given previously. If $N = N_D$ is the drastic fuzzy negation, then $I_{U',N}$ is given by the least fuzzy implication

$$I_{U',N}(x, y) = I_0(x, y) = \begin{cases} 1 & \text{if } x = 0 \text{ or } y = 1 \\ 0 & \text{otherwise,} \end{cases}$$

satisfies (1) *with respect to any conjunctive uninorm U.*

If N is continuous (the most usual case), uninorms in $\mathcal{U}_{\cos,\max}$ can be also discarded.

Proposition

$U \equiv \langle T, e, S\rangle$ conjunctive, $U' \equiv \langle T', e', S'\rangle$ disjunctive. N continuous fuzzy negation and $I_{U',N}$ its (U, N)-implication. If $I_{U',N}$ satisfies (1) with respect to U then U' can not be in $\mathcal{U}_{cos,\max}$.

When N is continuous, it must have a fixed point: e_N.

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- *If $e' < e_N$, then it must be $e_N < e$ and so $e' < e_N < e$.*

U-Modus Ponens for (U, N)-implications

General results

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- *If $e' < e_N$, then it must be $e_N < e$ and so $e' < e_N < e$.*
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Case $e' > e_N$ is not included because, initially, there is no restriction on the relative position of the neutral element e with respect to $e' > e_N$.

Case U' idempotent

- We will deal with the case when U' is idempotent, with $g'(0) = 1$ and $g'(1) = 0$.

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$U \equiv \langle T, e, S \rangle_{\min}$ and $U' \equiv \langle g', e' \rangle_{ide}$ disjunctive with $g'(0) = 1$ and $g'(1) = 0$. Take the fuzzy negation $N = g'$ and let $I_{U',N}$ be the corresponding (U, N)-implication. Then $I_{U',N}$ atisfies the U-Modus Ponens with respect to U if and only if the following items hold:

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Conclusions and future work

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- The case of idempotent uninorms has been started with some new results.

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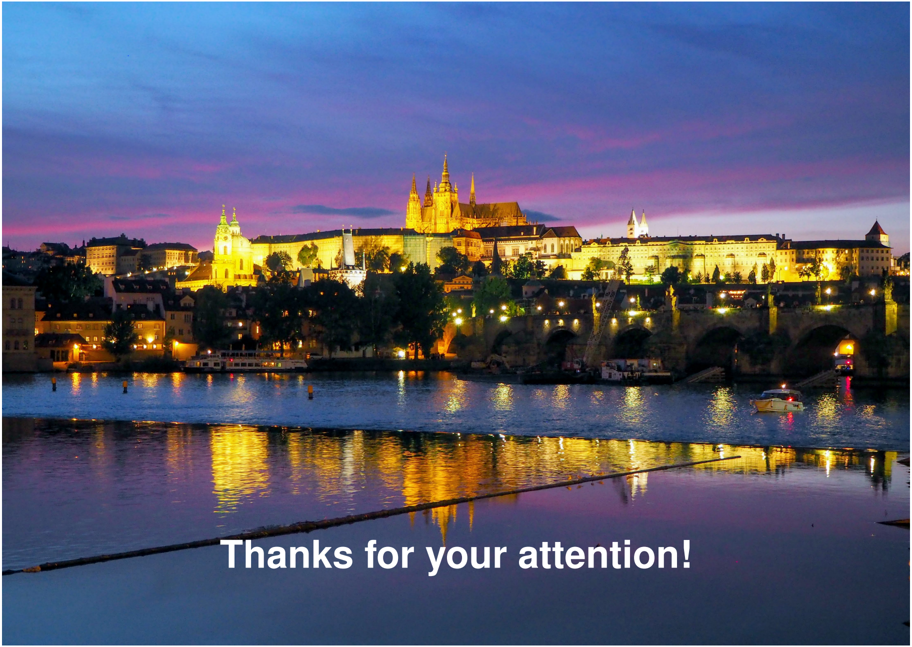
- Deal also with other kinds of implications like h and (h, e)-implications recently introduced by Massanet and Torrens (2011).


Thanks for your attention!