First steps towards the characterization of (S, N)-implications with a non-continuous fuzzy negation

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Introduction

Fuzzy implication functions

The definition of fuzzy implication function is enough flexible to allow the existence of a huge number of fuzzy implication functions.

Definition

A binary operation $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication function* if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Additional properties

These operators can satisfy additional properties that come usually from tautologies in classical logic.

1. *Exchange Principle*:

$$I(x, I(y, z)) = I(y, I(x, z)), \quad \text{for all } x, y, z \in [0, 1]. \qquad \textbf{(EP)}$$

2. *Left-neutrality principle*:

$$I(1, y) = y \qquad \text{for all} \quad y \in [0, 1]. \qquad \textbf{(NP)}$$

3. *Law of left contraposition* with respect to a fuzzy negation N:

$$I(N(x), y) = I(N(y), x), \quad \text{for all} \quad x, y \in [0, 1]. \qquad \textbf{(L-CP(N))}$$

4. *Law of right contraposition* with respect to a fuzzy negation N:

$$I(x, N(y)) = I(y, N(x)), \quad \text{for all} \quad x, y \in [0, 1]. \qquad \textbf{(R-CP(N))}$$

The need of characterization

From time to time, “new” families of fuzzy implication functions appear. However, some time later some of them are proved to have intersection with other old families or even they are actually the same family!

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From time to time, "new" families of fuzzy implication functions appear. However, some time later some of them are proved to have intersection with other old families or even they are actually the same family!

Solution

To axiomatically characterize the families of fuzzy implication functions in order to know better their structure and behavior.

The problem of the characterization of (S, N)-implications with a non-continuous negation

Although many families of fuzzy implication functions have already been characterized, there are others whose characterization remain unknown. For instance, in

Michał Baczyński and Balasubramaniam Jayaram. On the characterizations of (S, N)-implications. *Fuzzy Sets and Systems*, 158:1713-1727, 2007.

the following problem is posed

Problem

What is the characterization of (S, N)-implications generated from non-continuous negations?

Characterization of (S, N)-implications with a continuous fuzzy negation

Definition of (S, N)-implications

Definition

A binary operator $I : [0,1]^2 \to [0,1]$ is called an (S, N)*-implication* if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \quad x, y \in [0, 1].$$

Proposition

If $I_{S,N}$ is a (S, N)-implication, then

(i) $I_{S,N}$ satisfies (**NP**) and (**EP**).

(ii) $I_{S,N}$ satisfies **(R-CP(N))**.

Characterization of (S, N)-implications with N continuous

Theorem (Baczyński and Jayaram, 2007)

For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) I is an (S, N)-implication generated from some t-conorm S and some continuous fuzzy negation N.

(ii) I satisfies $(\mathbf{I_1})$, $(\mathbf{EP})$ and $I(x, 0)$ is a continuous fuzzy negation.

Moreover, the representation of the (S, N)-implication is unique with

$$N(x) = I(x, 0), \quad x \in [0,1],$$

$$S(x, y) = I(\mathfrak{R}_N(x), y), \quad x, y \in [0,1].$$

Modified pseudo-inverse of a fuzzy negation

Definition (Baczyński and Jayaram, 2007)

Let N be a continuous fuzzy negation and consider the function $\mathfrak{R}_N : [0,1] \to [0,1]$ defined by

$$\mathfrak{R}_N(x) = \begin{cases} N^{(-1)}(x) & \text{if} \quad x \in (0,1], \\ 1 & \text{if} \quad x = 0, \end{cases}$$

where $N^{(-1)}$ is the pseudo-inverse of N, i.e,

$$N^{(-1)}(y) = \sup\{x \in [0,1] \mid N(x) > y\} \text{ for } y \in [0,1].$$

Proposition

(i) $\mathfrak{R}_N$ is a fuzzy negation.

(ii) $\mathfrak{R}_N$ is a strictly decreasing function.

(iii) $N \circ \mathfrak{R}_N = \mathrm{id}_{[0,1]}$.

(iv) $\mathfrak{R}_N \circ N|_{\mathrm{Ran}(\mathfrak{R}_N)} = \mathrm{id}_{[0,1]}|_{\mathrm{Ran}(\mathfrak{R}_N)}$.

Towards the characterization of (S, N)-implications with a non-continuous fuzzy negation

Problem 1: Properties of $\mathfrak{R}_N$ when N is non-continuous

If N is a non-continuous negation then $\mathfrak{R}_N$ is still a fuzzy negation but

- $\mathfrak{R}_N$ is not a strictly decreasing function.
- The equalities $N \circ \mathfrak{R}_N = \mathrm{id}_{[0,1]}$ and $\mathfrak{R}_N \circ N|_{\mathrm{Ran}(\mathfrak{R}_N)} = \mathrm{id}_{[0,1]}|_{\mathrm{Ran}(\mathfrak{R}_N)}$ are not necessarily satisfied.

Result 1

If N is continuous then

$$\mathfrak{R}_N \circ N|_{\text{Ran}(\mathfrak{R}_N)} = \text{id}_{[0,1]}|_{\text{Ran}(\mathfrak{R}_N)}$$

If N is non-continuous we have the following result

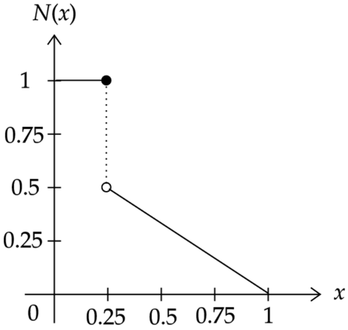
Proposition

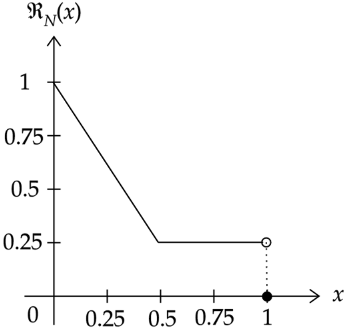
Let $N : [0,1] \to [0,1]$ be a fuzzy negation and $x_0 \in [0,1]$, then

$$\mathfrak{R}_N \circ N(x_0) \neq x_0 \quad \Leftrightarrow \quad \begin{array}{l} x_0 \notin \text{Ran}(\mathfrak{R}_N) \text{ or } x_0 \in \text{Ran}(\mathfrak{R}_N) \text{ and } \exists \varepsilon > 0 \text{ such that} \\ N|_{(x_0-\varepsilon, x_0]} \text{ is a constant function and} \\ \lim\limits_{x \to x_0^+} N(x) \neq N(x_0). \end{array}$$

Result 1

Example





$$\Re_N \circ N(x) = x, \quad x \in \mathrm{Ran}(\Re_N) - \{0.25\}.$$

Result 2

If N is continuous then

$$N \circ \mathfrak{R}_N = \mathrm{id}_{[0,1]}.$$

If N is non-continuous we have the following result

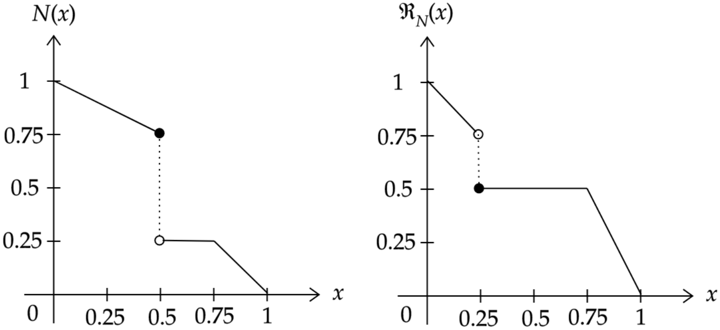
Proposition

Let $N : [0,1] \to [0,1]$ be a decreasing non-constant function and $y_0 \in [0,1]$, then

$$N \circ \mathfrak{R}_N(y_0) \neq y_0 \quad \Leftrightarrow \quad \begin{array}{l} y_0 \notin \mathrm{Ran}(N) \text{ or } y_0 \in \mathrm{Ran}(N) \text{ and } \exists x_0 \in [0,1] \\ \text{ such that } N(x_0) = y_0 \text{ and } \exists \varepsilon > 0 \text{ such that} \\ N|_{(x_0-\varepsilon, x_0]} \text{ is a constant function and} \\ \displaystyle\lim_{x \to (x_0-\varepsilon)^+} N(x) = N(x_0) \neq N(x_0 - \varepsilon). \end{array}$$

Result 2

Example



$$N \circ \Re_N(y) = y, \quad y \in \mathrm{Ran}(N) - \{0.25\}.$$

Problem 2: Non-unicity of S

Example

Let us consider the following non-continuous fuzzy negation

$$N(x) = \begin{cases} 1 - x & \text{if } x \in [0, 0.25] \cup [0.75, 1], \\ 0.25 & \text{if } x \in (0.25, 0.75). \end{cases}$$

and the following continuous t-conorms

$$S_M(x, y) = \max(x, y), \quad (x, y) \in [0, 1]^2,$$

$$S(x, y) = \begin{cases} 2xy + 0.5x + 0.5y - 0.125 & \text{if } x, y \in (0.25, 0.75), \\ \max(x, y) & \text{Otherwise.} \end{cases}$$

Then,

$$I_{S_M,N}(x, y) = I_{S,N}(x, y) = \begin{cases} \max(0.25, y) & \text{if } x, y \in (0.25, 0.75), \\ max(1 - x, y) & \text{Otherwise.} \end{cases}$$

Problem 2: Non-unicity of S

Proposition

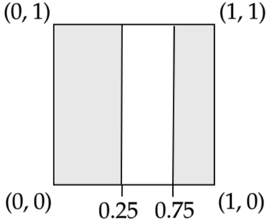
Let N_1, N_2 be two fuzzy negations and S_1, S_2 two t-conorms. Then,

$$\begin{array}{c} I_{S_1,N_1}(x,y) = I_{S_2,N_2}(x,y) \\ (x,y) \in [0,1]^2. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} N_1(x) = N_2(x) = N(x), & x \in [0,1], \\ S_1(x,y) = S_2(x,y), & (x,y) \in \mathrm{Ran}(N) \times [0,1]. \end{array}$$

Problem 2: Non-unicity of S

Example (cont.)

The only values of the t-conorms that used for the definition of the (S, N)-implication are:



Problem 2: Non-unicity of S

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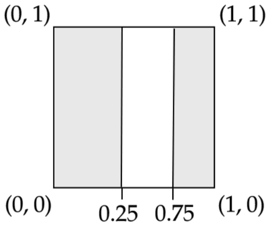


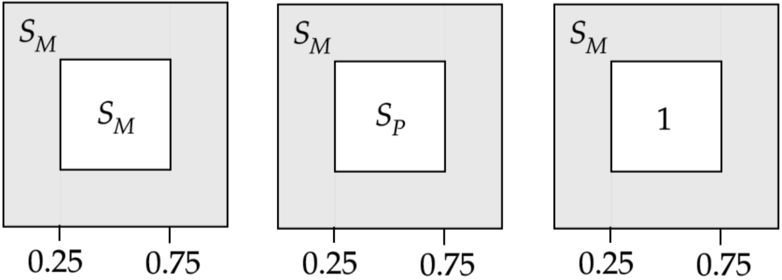
Moreover, even when we consider a function that is not a t-conorm

$$F(x) = \begin{cases} 1 & x, y \in (0.25, 0.75), \\ \max(x, y) & \text{Otherwise}, \end{cases}$$

we can have that

$$I_{S_M,N}(x, y) = I_{S,N}(x, y) = I_{F,N}(x, y), \quad (x, y) \in [0, 1]^2.$$

Problem 2: Non-unicity of S



Equivalence relation

Definition

Let N be a fuzzy negation and S_1, S_2 two t-conorms. Then we define the relation $\equiv_N$ as

$$S_1 \equiv_N S_2 \Leftrightarrow I_{S_1,N}(x,y) = I_{S_2,N}(x,y), \quad (x,y) \in [0,1]^2.$$

Lemma

- $\equiv_N$ is an equivalence relation.
- If N is continuous then $S_1 \equiv_N S_2 \Leftrightarrow S_1(x,y) = S_2(x,y), \quad (x,y) \in [0,1]^2$.

Problem 3: Construction of a representative

Definition

Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function and N a fuzzy negation. Let us define a function $S_{I,N} : A \times [0,1] \to [0,1]$ as follows:

$$S_{I,N}(x,y) = I(\mathfrak{R}_N(x), y), \quad (x,y) \in A \times [0,1],$$

where $A = \{x \in \mathrm{Ran}(N) \mid N \circ \mathfrak{R}_N(x) = x\}$.

Properties of $S_{I,N}$

Proposition

Let $I : [0,1]^2 \to [0,1]$ be a fuzzy implication function and N a fuzzy negation with only one point of discontinuity. Then

(i) $S_{I,N}(\bullet, y)$ is increasing in $x \in A$ for $y \in [0,1] \Leftrightarrow I(\bullet, y)$ is decreasing in $x \in \mathrm{Ran}(\mathfrak{R}_N)$ for $y \in [0,1]$.

(ii) $S_{I,N}(x, \bullet)$ is increasing in $y \in [0,1]$ for $x \in A \Leftrightarrow I(x, \bullet)$ is increasing in $y \in [0,1]$ for $x \in \mathrm{Ran}(\mathfrak{R}_N)$.

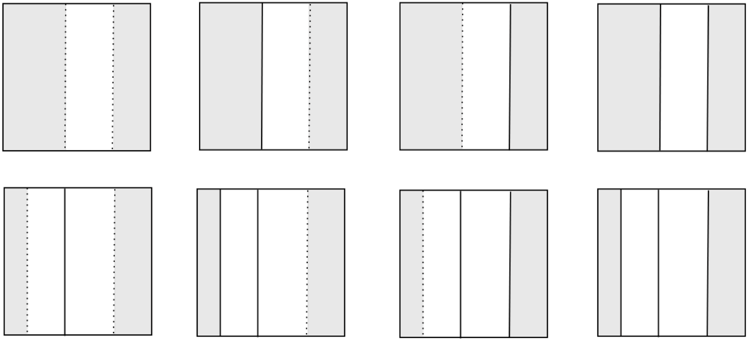
(iii) $S_{I,N}(x,y) = S_{I,N}(y,x)$ for $x, y \in A \Leftrightarrow I$ satisfies L-CP($\mathfrak{R}_N$) for $x, y \in A$.

(iv) $S_{I,N}(x,0) = x$ for $x \in A \Leftrightarrow I(1,x) = x$ for $x \in A$.

(v) $S_{I,N}(x, S_{I,N}(y,z)) = S_{I,N}(y, S_{I,N}(x,z))$ for $x, y \in A$ and $z \in [0,1] \Leftrightarrow$ $I(x, I(y,z)) = I(y, I(x,z))$ for $x, y \in \mathrm{Ran}(\mathfrak{R}_N)$ and $z \in [0,1]$.

Problem 3: Construction of a representative

In order to find a representative of the corresponding class of t-conorms, we need to find the extension of $S_{I,N}$ to $[0,1]^2$.

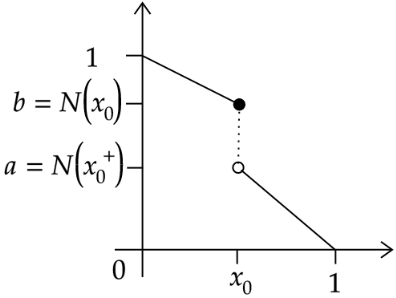


Characterization of (S, N)-implications with a fuzzy negation with only one point of discontinuity and continuous t-conorms in the class of the maximum t-conorm

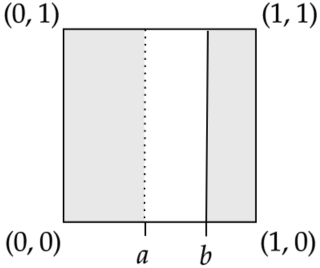
Equivalence class of S_M

Let N be a fuzzy negation with only one point of discontinuity x_0. Depending on the type of discontinuity we have different cases:

1. If N is such that

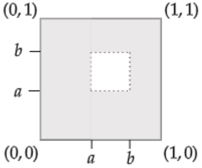


then, the class $[S_M]_N$ consist of t-conorms that are equal to S_M in the following region



Equivalence class of S_M

By the commutativity of the t-conorms and by continuity we can consider t-conorms that are equal to S_M in the following region



Now, by the representation theorem of continuous t-conorms we obtain the following result.

Proposition

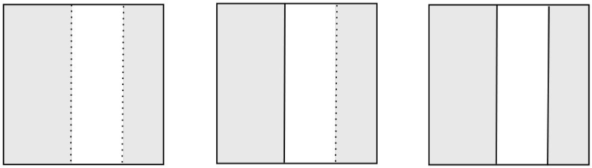
Let N be a fuzzy negation with one point of discontinuity x_0 such that $N(x_0) = N(x_0^-) \neq N(x_0^+)$. Then,

$$S \equiv_N S_M \Leftrightarrow S(x,y) = \begin{cases} a + (b-a) \cdot \tilde{S}\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right) & \text{if} \quad (x,y) \in [a,b]^2, \\ \max(x,y) & \textit{Otherwise}, \end{cases}$$

where $a = N(x_0^+)$, $b = N(x_0)$ and $\tilde{S}$ is a continuous t-conorm.

Equivalence class of S_M

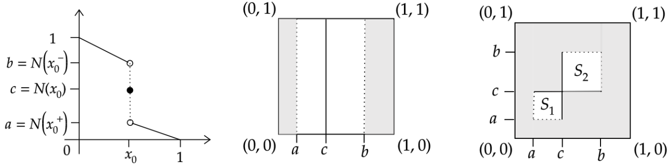
2. For any other N such that $N(x_0)$ coincides with $N(x_0^-)$ or $N(x_0^+)$, then the class $[S_M]_N$ consist of t-conorms that are equal to S_M in one of the following regions:



Analogously to the previous case we obtain the same result.

Equivalence class of S_M

3. For any other N such that $N(x_0)$ does not coincide with $N(x_0^-)$ or $N(x_0^+)$ we have that



Proposition

Let N be a fuzzy negation with one point of discontinuity x_0 such that $N(x_0) = N(x_0^+) \neq N(x_0^-)$. Then,

$$S \equiv_N S_M \Leftrightarrow S(x,y) = \begin{cases} a + (c-a)\cdot S_1\left(\frac{x-a}{c-a}, \frac{y-a}{c-a}\right) & \text{if} \quad (x,y) \in [a,c]^2, \\ c + (b-c)\cdot S_2\left(\frac{x-c}{b-c}, \frac{y-c}{b-c}\right) & \text{if} \quad (x,y) \in [c,b]^2, \\ \max(x,y) & \textit{Otherwise}, \end{cases}$$

where $a = N(x_0^+)$, $c = N(x_0)$, $b = N(x_0^-)$ and S_1, S_2 are two continuous t-conorms.

Characterization

Theorem

For a function $I : [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) I is an (S, N)-implication with a negation N with one point of discontinuity and a continuous t-conorm $S \in [S_M]_N$.

(ii) I satisfies:

1. N_I is a fuzzy negation with one point of discontinuity.
2. $I(x, I(y, z)) = I(y, I(x, z))$ for $x, y \in \mathrm{Ran}(\mathfrak{R}_I)$ and $z \in [0,1]$.
3. $I(x_1, y) \geq I(x_2, y)$ when $x_1, x_2 \in \mathrm{Ran}(\mathfrak{R}_I)$ and $x_1 \leq x_2$ for all $y \in [0,1]$.
4. $I(x, y_1) \leq I(x, y_2)$ when $y_1, y_2 \in [0,1]$ and $y_1 \leq y_2$ for all $x \in A$.
5. If $N_I(x) = N_I(\tilde{x})$ for $\tilde{x} \in [x, x+\varepsilon)$ and $\varepsilon > 0 \Rightarrow I(x, y) = I(\tilde{x}, y)$ for $y \in [0,1]$.
6. $I(\mathfrak{R}_I(x), x) = x$ for $x \in A$.

In this case $N(x) = I(x, 0)$.

Conclusions and future Work

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In this sense, we have provided some steps to follow in order to achieve the characterization.

Although most of our results are valid in a more general situation, we provide a complete characterization in the case when N is a fuzzy negation with one point of discontinuity and S is the maximum t-conorm.

Future work

What remains to be solved to characterize this class of implications:

- The case in which S is any continuous t-conorm.

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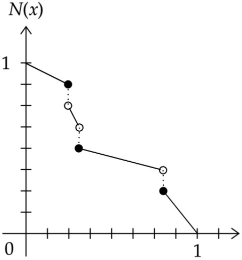
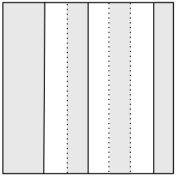
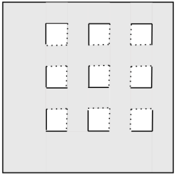
- The case in which S is any continuous t-conorm.
- Consider negations with more than one point of discontinuity.







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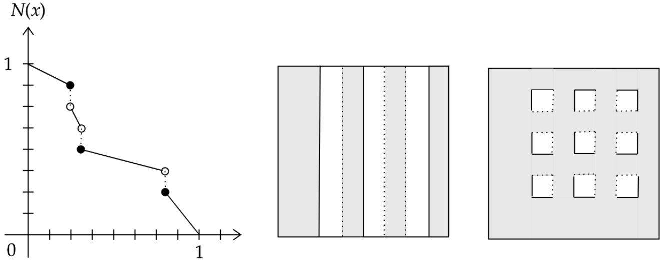
- The case in which S is any continuous t-conorm.
- Consider negations with more than one point of discontinuity.



- The case in which S is a non-continuous t-conorm.

Thank you for your attention!