Graphical Characterization of Discrete RU and (U,N)-Implications Derived from Idempotent Uninorms

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Introduction

Fuzzy logics with a finite number of truth values are a great tool to modelize linguistic opinions given by experts. The truth values can be always medelled by a discrete chain.

Very Bad	Bad	Good	Very Good
↓	↓	↓	↓
0	1	2	3

We focused on the study of some discrete fuzzy operators: discrete uninorms and some derived families of fuzzy implication functions.

Some notation and definitions

- $L_n = \{0, 1, ..., n\}$
- $F : L_n^2 \to L_n$ is idempotent if and only if $F(x, x) = x$ for all $x \in L_n$.
- $F : L_n^2 \to L_n$ is conservative if and only if $F(x, y) \in \{x, y\}$ for all $(x, y) \in L_n^2$.

Uninorms

Definition

A discrete uninorm is a function $U : L_n^2 \to L_n$ which is:

- Associative.
- Commutative.
- Increasing in each variable.
- Has a neutral element $x \in L_n$.

Characterization of Idempotent Uninorms

The following results were presented in [1]

Definition

A linear ordering $\preceq$ on L_n is said to be *single-peaked* if for any $a, b, c \in L_n$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$.

[1] M. Couceiro, J. Devillet, and J.-L. Marichal, "Characterizations of idempotent discrete uninorms," Fuzzy Sets and Systems, vol. 334, pp. 60–72, 2018.

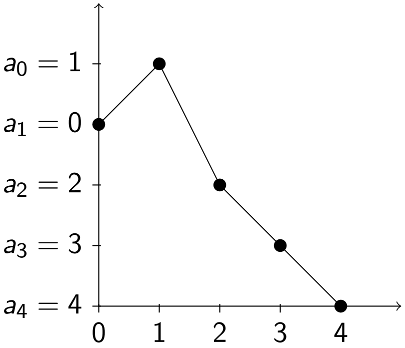


Figure: Representation of $1 \prec 0 \prec 2 \prec 3 \prec 4$ and $3 \prec 1 \prec 2 \prec 4 \prec 0$.

Theorem

A function $F : L_n^2 \to L_n$ is an idempotent discrete uninorm if and only if there exists a single-peaked linear ordering $\preceq$ such that $F = \max_{\preceq}$ where $\max_{\preceq}$ denotes the maximum operation with respect to $\preceq$.

- Idempotent uninorms are conservative.

RU and (U,N)-implications

Definition

A discrete implication is a function $I : L_n^2 \to L_n$ which is:

- Decreasing in the first component.
- Increasing in the second component.
- $I(0,0) = I(n,n) = n$ and $I(n,0) = 0$.

Definition

Given a conjunctive uninorm $U : L_n^2 \to L_n$ the binary operator

$$I_U(x,y) = \max\{z \in L_n \mid U(x,z) \leq y\}, \ x,y \in L_n$$

is called a RU-implication

- U is conjuctive if $U(n,0) = 0$.

Definition

Given a disjuctive uninorm $U : L_n^2 \to L_n$ the binary operator

$$I_{U,N_c}(x, y) = U(n - x, y)$$

is called a (U, N_c)-implication.

- U is disjunctive if $U(n, 0) = n$.

Known Characterizations

This result was presented in [2]

Proposition

Given $I : L_n^2 \to L_n$ a function and $e \in L_n$ with $e > 0$. The following statements are equivalent:

(1) *I is a RU-implication derived from a discrete idempotent uninorm with neutral element e.*

(2) *I satisfies*

- *Increasing in the second component.*
- *Exchange principle (EP) and property (OP_e).*
- *$I(x, x) \geq x$ for each $x \geq e$.*
- *$I(x, x-1) < x$ for each $0 < x < e$.*

- (EP): $I(x, I(y, z)) = I(y, I(x, z))$.
- (OP_e): $I(x, y) \geq e \Leftrightarrow x \leq y$.

[2] M.Mas, G.Mayor,M.Monserrat, and J. Torrens, "Residual implications from discrete uninorms. a characterization," in Enric Trillas: A Passion for Fuzzy Sets: A Collection of RecentWorks on Fuzzy Logic, L.Magdalena, J. L. Verdegay, and F. Esteva, Eds. Cham: Springer International Publishing, 2015, pp. 27–40.

The following characterization is new but it is analogous to the $[0,1]$ case presented in[3].

Theorem

Given $I : L_n^2 \to L_n$ a function. The following statements are equivalent:

(1) *I is a (U, N_c)-implication generated by a discrete idempotent uninorm with neutral element $0 < e < n$.*

(2) *I satisfies*
   - (a) $I(x,z) \geq I(y,z)$ *for each* $x, y, z \in L_n$ *such that* $x \leq y$.
   - (b) $I(0,0) = n$.
   - (c) (EP).
   - (d) $I(x,e) = n - x$ *for some* $e \in]0, n[$.
   - (e) $I(n-x, x) = x$ *for each* $x \in L_n$.

- (EP): $I(x, I(y,z)) = I(y, I(x,z))$.

[3] M. Baczynski and B. Jayaram, "(U, N)-implications and their characterizations," Fuzzy Sets and Systems, vol. 160, no. 14, pp. 2049–2062, 2009.

New Characterizations

Theorem

A function $I : L_n^2 \to L_n$ is a RU-implication derived from a discrete idempotent uninorm if and only if there exists a single-peaked linear ordering $\preceq$ with $n \preceq 0$ such that:

$$I(x,y) = \max\{z \in L_n \mid \max_{\preceq}(x,z) \leq y\}$$
$$= \begin{cases} \max\{z \in [0,y] \mid z \succ x\} & \text{if } x > y, \\ \max\{z \in [e,n] \mid z \leq y \text{ or } z \preceq x\} & \text{if } x \leq y. \end{cases}$$

Theorem

A function $I : L_n^2 \to L_n$ *is a* (U, N_c)*-implication if and only if there exists a single-peaked linear ordering* $\preceq$ *such that* $I(x, y) = \max_{\preceq}(n - x, y)$.

Contour plot

The contour plot of a function $F : L_n^2 \to L_n$ is a graphical representation of the common values of the function.

Example

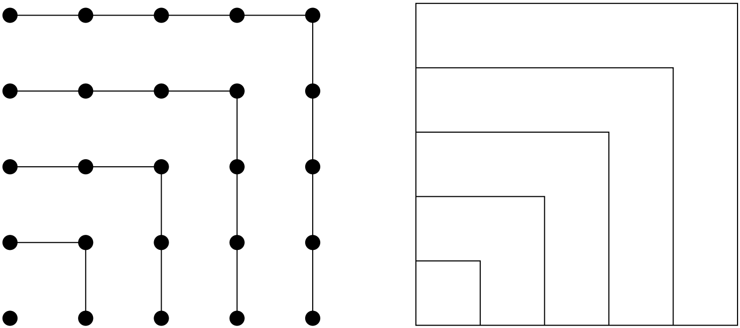


Figure: Contour plot of $\max(x, y)$ on L_4^2.

Algorithms

- With the results obtained we can identify each uninorm with a single-peaked ordering and each RU and (U, N_c)-implication with the idempotent uninorm that derives it.
- With this information in mind, we made algorithms with different utilities.

Algorithms

1. Generator of single-peaked orderings.
2. Ordering from an idempotent uninorm.
3. Ordering from an idempotent uninorm given the RU or (U, N_c)-implication.
4. Contour plot of an idempotent uninorm given the associated ordering.[4]
5. Contour plot of a (U, N_c)-implication given the associated ordering.
6. (Claim) Contour plot of a RU-implication given the associated ordering.

[4] M. Couceiro, J. Devillet, and J.-L. Marichal, "Characterizations of idempotent discrete uninorms," Fuzzy Sets and Systems, vol. 334, pp. 60–72, 2018.

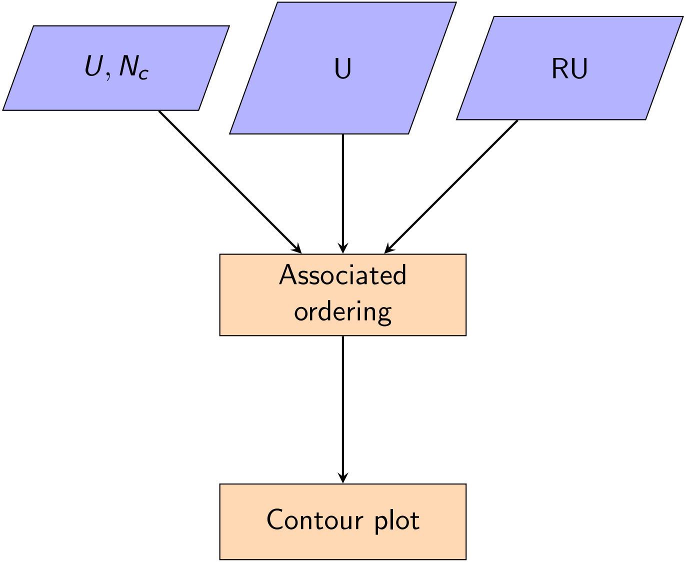


Figure: Algorithms diagram.

Algorithm 1: Ordering from an idempotent uninorm

Input: Idempotent uninorm $U : L_n^2 \to L_n$
Output: Linear ordering associated U

$x = 0, y = n$;
for $i \leftarrow n$ **to** 1 **by** -1 **do**
  $a_i = U(x, y)$;
  if $a_i == x$ **then**
    $x = x + 1$
  else
    $y = y - 1$
$a_0 = x$;
return $\{a_0, \ldots, a_n\}$

- The algorithm compares the values of L_n to see how are they ordered.

Example

We have an idempotent uninorm $U : L_8^2 \to L_8$ given by:

$$U(x,y) = \begin{cases} \min(x,y) & \textit{if } \; x+y \leq 8, \\ \max(x,y) & \textit{if } \; x+y > 8. \end{cases}$$

Using the algorithm we obtain the associated ordering:

$$4 \prec 5 \prec 3 \prec 6 \prec 2 \prec 7 \prec 1 \prec 8 \prec 0$$

With the associated ordering we can make the contour plot.

Algorithm 2: Contour plot of an idempotent uninorm.

Input: Single-peaked ordering $ORD = \{a_0, ..., a_n\}$
Output: Contour plot of U

```
Set C_0 = {a_0};
for k ← 1 to n do
    C_k = {a_k} ∪ C_{k-1};
    Connect the pairs of points (x, y), (u, v) from C_k^2 \ C_{k-1}^2 with
     x = u or y = v closer to each other, these points have value a_k ;
```

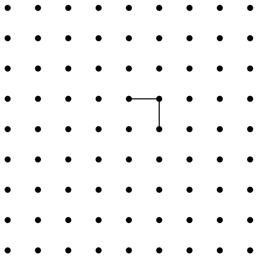
```
C_0={4}

k=1

C_1={4,5}

C_1^2\C_0^2={(4,5),(5,5),(5,4)}

Connect:
(4,5) with (5,5)
(5,5) with (5,4)
```

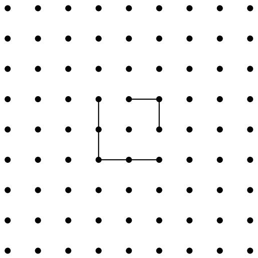


```
k=2

C_2={4,5,3}

C_2^2\C_1^2={(3,5),(3,4),(3,3),(4,3),(5,3)}

Connect:
(3,5) with (3,4)
(3,4) with (3,3)
(3,3) with (4,3)
(4,3) with (5,3)
```



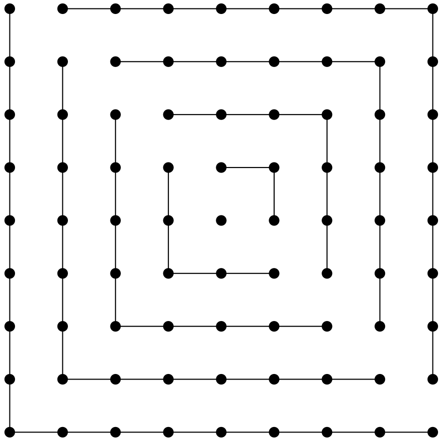


Figure: Contour plot of U.

- If we have a RU or (U, N_c)-implication we could do a similar process and obtain the ordering, the idempotent uninorm and the contour plots.

A brief explanation of the contour plot of a RU-implication algorithm.

Main ideas:

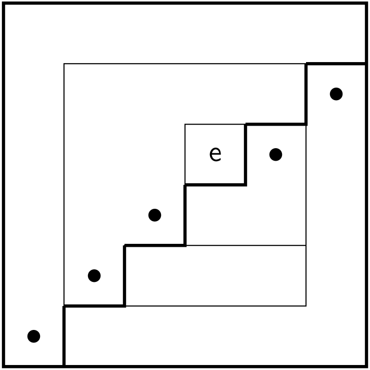
- RU-implications satisfy the (OP_e) property, as a consequence we can divide the contour plot in two pieces lower and upper triangle.
- The structure of the RU-implicaction is a composition of pilars.

Definition

Given I an RU-implication. We name *pilar of the lower triangle* (resp. upper) of a contour plot of pilars of I to each polygon that has a point (x, y) with $x - 1 = y$ and $x > e$ (resp $x = y$ and $x < e$). We call *width* of the pilar to the number of points of this kind that it contains.

Example

We can see the points in the RU-implication with associated ordering $3 \prec 4 \prec 2 \prec 1 \prec 5 \prec 0$:



Once we know the width of each pilar (not proven) we can build the contour plot as follows:

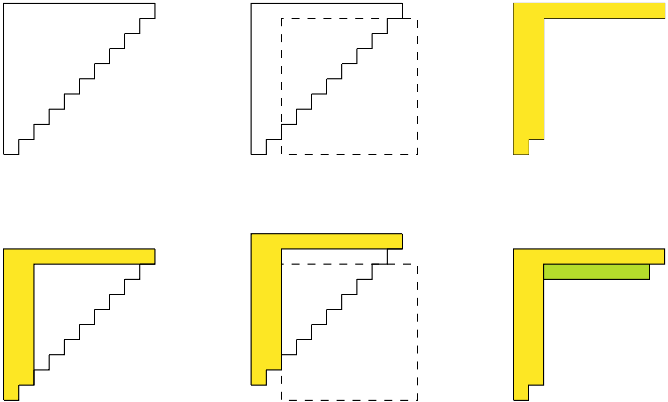


Figure: Two steps of the process to build the contour plot of the upper triangle of the RU-implication with associated ordering $ORD = \{5, 4, 6, 7, 3, 2, 8, 9, 1, 0\}$.

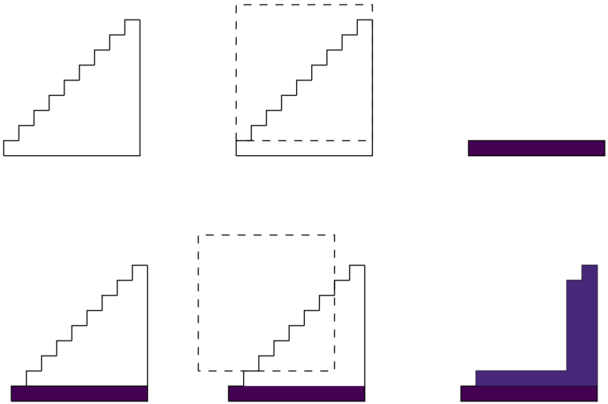
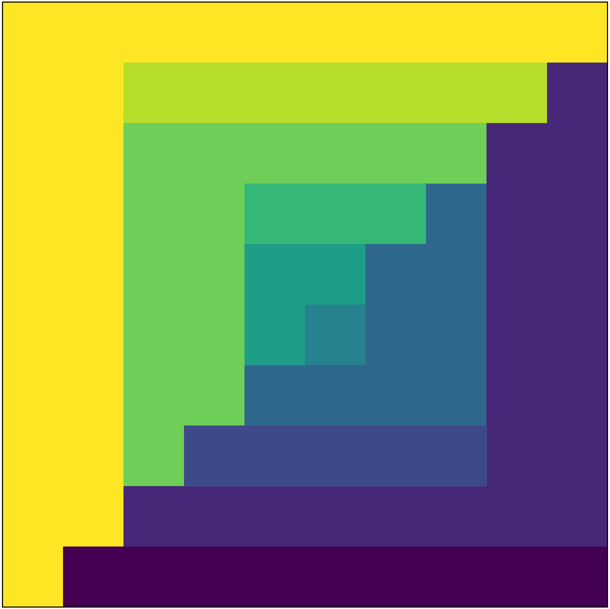


Figure: Two steps of the process to build the contour plot of the lower triangle of the RU-implication with associated ordering $ORD = \{5, 4, 6, 7, 3, 2, 8, 9, 1, 0\}$.



Some conclusions

With this work we have

- Obtained new characterizations for RU and (U, N_c)-implications.
- Created algorithms to
  - Obtain the associated ordering of uninorms and implications derived.
  - Make the contour plot of some functions.

Future work:

- Prove the correctness of the last algorithm.
- Expand this work to other families of functions.

THE END