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Some remarks about polynomial aggregation functions

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Soft Computing, Image Processing and Aggregation

Main points of this talk

1. Motivation
   - Aggregation functions
2. Preliminaries
   - Aggregation functions: basic properties of binary aggregation functions
3. Polynomial Binary Aggregation Functions with a non trivial 0 region
   - Degree 0 and 1
   - Degree 2
4. Polynomial Binary Aggregation Functions with a non trivial 1-region
5. Conclusions and future work

Motivation

Motivation

- Aggregation functions play a fundamental role in social and scientific sciences because in all of them it becomes necessary at some point to merge several input data into a representative output value.
- This is the main reason for which the theoretical study of aggregation functions have experienced an important growth in last decades.

Motivation

- Taking in mind the applications (for instance, decision making process, image processing, approximate reasoning, ...), one can look for aggregation functions having expressions as simple as possible.
- A first step could be to study aggregation functions whose expression is given by polynomial or rational functions of different degrees.

Motivation

This was already done in some particular cases.

- All the rational Archimedean continuous t-norms are characterized as the well-known Hamacher class which contains the t-norms given by the following expression

$$T_\alpha(x,y) = \frac{xy}{\alpha + (1-\alpha)(x+y-xy)}, \quad x,y \in [0,1]$$

  with $\alpha \geq 0$

- All the rational uninorms were characterized as those whose expression is given by

$$U_e(x,y) = \frac{(1-e)xy}{(1-e)xy + e(1-x)(1-y)}$$

  if $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and, $U(1,0) = U(0,1) = 0$ or $U(0,1) = U(1,0) = 1$.

Motivation and Main Goal

- A similar study for polynomial and rational fuzzy implication functions have been proposed.
- Recently, an initial study of aggregation functions whose expressions are given by polynomial functions was presented.
- Now, we want to investigate on binary aggregation functions given by polynomial expressions only in a particular sub-domain of the unit interval.

Preliminaries

Definition

An *n-ary aggregation function* is a function of $n > 1$ arguments that maps the (n-dimensional) unit cube onto the unit interval $f : [0,1]^n \to [0,1]$, with the properties

i) $f(\underbrace{0, 0, \ldots, 0}_{n-\text{times}}) = 0$ and $f(\underbrace{1, 1, \ldots, 1}_{n-\text{times}}) = 1$.

ii) $\mathbf{x} \leq \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$.

In particular, a 2-ary aggregation function will be called a *binary aggregation function*.

Properties of binary aggregation functions

- The *idempotency*,

$$f(x,x) = x, \quad \text{for all } x \in [0,1]. \tag{\textbf{ID}}$$

- The *symmetry*,

$$f(x,y) = f(y,x), \tag{\textbf{SYM}}$$

  for all $(x,y) \in [0,1]^2$.
- The *associativity*,

$$f(f(x,y),z) = f(x,f(y,z)), \quad \text{for all } x,y,z \in [0,1]. \tag{\textbf{ASS}}$$

Properties of binary aggregation functions

- It is said that $a \in (0, 1)$ is a *zero divisor* when

$$f(a, y) = f(y, a) = 0, \qquad (\mathbf{ZD}(\mathbf{a}))$$

  for some $y > 0$.
- It is said that $a \in (0, 1)$ is a *one divisor* when

$$f(a, y) = f(y, a) = 1, \qquad (\mathbf{OD}(\mathbf{a}))$$

  for some $y < 1$.

Properties of binary aggregation functions

- The *left neutral element property* with a fixed $e \in [0, 1]$,

$$f(e, y) = y, \quad \text{for all } y \in [0, 1]. \qquad (\mathbf{L - NE(e)})$$

- The *right neutral element property* with a fixed $e \in [0, 1]$,

$$f(x, e) = x, \quad \text{for all } x \in [0, 1]. \qquad (\mathbf{R - NE(e)})$$

- The *neutral element property* with a fixed $e \in [0, 1]$,

$$f(e, x) = f(x, e) = x, \quad \text{for all } x \in [0, 1]. \qquad (\mathbf{NE(e)})$$

Properties of binary aggregation functions

- The *left absorbing element property* with a fixed $a \in [0, 1]$,

$$f(a, y) = a, \quad \text{for all } y \in [0, 1]. \qquad (\mathbf{L} - \mathbf{AE}(\mathbf{a}))$$

- The *right absorbing element property* with a fixed $a \in [0, 1]$,

$$f(x, a) = a, \quad \text{for all } x \in [0, 1]. \qquad (\mathbf{R} - \mathbf{AE}(\mathbf{a}))$$

- The *absorbing element property* with a fixed $a \in [0, 1]$,

$$f(a, x) = f(x, a) = a, \quad \text{for all } x \in [0, 1]. \qquad (\mathbf{AE}(\mathbf{a}))$$

Some additional concepts

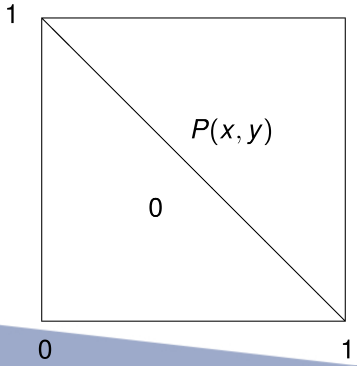
- We will call *Conjunctors* to those binary aggregation functions with absorbing element 0.
- We will call *Disjunctors* to those aggregation functions with absorbing element 1.
- Note that conjunctive (disjunctive) aggregation functions (those that take values below the minimum/over the maximum) are trivially conjunctors (disjunctors) but not vice versa.

Polynomial Binary Aggregation Functions with a non trivial 0 region

We will study functions of the form

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 1 - x, \\ P(x, y) & \text{if } y > 1 - x, \end{cases}$$

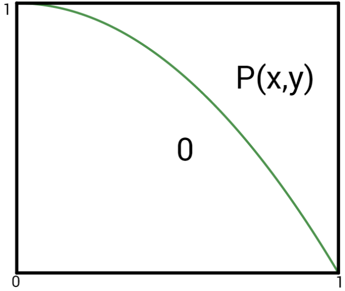
where $P(x, y)$ is a polynomial of degree 1 or 2.



Another possibility is to investigate functions of the form

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1 - x, \\ P(x,y) & \text{if } y > 1 - x, \end{cases}$$

where $P(x, y)$ is a polynomial of degree 1 or 2.
In this case we use $N(x) = 1 - x^2$.



Polynomial Binary Aggregation Functions with a non trivial 0 region

Example

(i) If we take as $P(x, y)$ any binary polynomial aggregation function of degree one or two we trivially obtain aggregation functions of the desired form.

(ii) The Łukasiewicz t-norm $T_{\mathbf{L}}$ is also an aggregation function of the previous form, just taking as $P(x, y)$ the polynomial $P(x, y) = x + y - 1$.

(iii) There are examples with $P(x, y)$ of any degree. Take for instance

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 1 - x, \\ \frac{x^n + y^n}{2} & \text{if } y > 1 - x. \end{cases}$$

Degree 0 and 1

Degree 0 and 1

Theorem

Let f be a binary function of the form

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ ax+by+c & \text{if } y > 1-x, \end{cases}$$

The following statements are equivalent:

i) *f is a binary aggregation function.*

ii) *There exist $a, b \in [0,1]$ such that $f = f_{a,b}$ where $f_{a,b} : [0,1]^2 \to [0,1]$ is given by*

$$f_{a,b}(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ ax+by+1-a-b & \text{if } y > 1-x. \end{cases} \tag{1}$$

Example

(i) The case $a = b = 0$ gives the only aggregation function of the form

$$f_{0,0}(x, y) = \begin{cases} 0 & \text{if } y \leq 1 - x, \\ 1 & \text{if } y > 1 - x. \end{cases}$$

(ii) The first and second projections with a 0 region limited by N_c are also obtained in the cases $a = 1, b = 0$, and $a = 0, b = 1$, respectively.

Example

(iii) The Łukasiewicz t-norm is obtained taking $a = b = 1$.

(iv) Taking $b = 1 - a$ we obtain the family of weighted arithmetic means with a 0 region limited by N_c, that is,

$$f_{a,1-a}(x, y) = \begin{cases} 0 & \text{if } y \leq 1 - x, \\ ax + (1 - a)y & \text{if } y > 1 - x. \end{cases} \tag{2}$$

It is easy to characterize all binary aggregation functions of the previous form fulfilling some additional properties. We collect some of them in the following proposition.

Proposition

The following statements are true:

(i) $f_{a,b}$ *is continuous if and only if* $a = b = 1$, *that is, if and only if* $f_{a,b} = f_{1,1}$ *is the Łukasiewicz t-norm.*

(ii) $f_{a,b}$ *satisfies* **(SYM)** *if and only if* $a = b$. *That is, when* $f_{a,b} = f_{a,a}$ *is given by*

$$f_{a,a}(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ a(x+y)+1-2a & \text{if } y > 1-x. \end{cases}$$

(iii) $f_{a,b}$ *satisfies* **(ID)** *in its positive region, if and only if* $b = 1-a$. *That is, when* $f_{a,b} = f_{a,1-a}$ *is given by*

$$f_{a,1-a}(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ ax+(1-a)y & \text{if } y > 1-x. \end{cases}$$

.

Proposition

(iv) $f_{a,b}$ *satisfies* **(L-NE(e))** *if and only if the neutral element is* $e = 1$ *and* $b = 1$*. That is, when* $f_{a,b} = f_{a,1}$ *is given by*

$$f_{a,1}(x, y) = \begin{cases} 0 & \textit{if } y \leq 1 - x, \\ ax + y - a & \textit{if } y > 1 - x. \end{cases}$$

(v) $f_{a,b}$ *satisfies* **(R-NE(e))** *if and only if and only if the neutral element is* $e = 1$ *and* $a = 1$*. That is, when* $f_{a,b} = f_{1,b}$ *is given by*

$$f_{1,b}(x, y) = \begin{cases} 0 & \textit{if } y \leq 1 - x, \\ x + by - b & \textit{if } y > 1 - x. \end{cases}$$

(vi) $f_{a,b}$ *satisfies* **(NE(e))** *if and only if the neutral element is* $e = 1$ *and* $f_{a,b} = f_{1,1}$ *is the Łukasiewicz t-norm.*

(vii) $f_{a,b}$ *satisfies* **(ASS)** *if and only if* $f_{a,b} = f_{1,1}$ *is the Łukasiewicz t-norm.*

Degree 2

Degree 2

Goal

We are interested in characterizing which functions of the form

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x \\ ax^2 + by^2 + cxy + dx + ey + f & \text{if } y > 1-x, \end{cases}$$

where a, b, c, d, e, f are real numbers such that $a^2 + b^2 + c^2 > 0$, are aggregation functions.

Example

(i) The family

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ by^2 + bxy + dx + ey + 1 - 2b - d - e & \text{if } y > 1-x, \end{cases}$$

where $-1 < b < 0$, $-b \leq d \leq 1-d$ and $-3b \leq e \leq 1-2b$ is a family of binary aggregation functions with a 0-region limited by N_c and given by a polynomial of degree 2 over N_c.

Example

(ii) The family

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ cxy + 2\sqrt{-c}x - cy + 1 - 2\sqrt{-c} & \text{if } y > 1-x, \end{cases}$$

where $-4 \leq c \leq -1$ is also a family of these aggregation functions which depends only on one parameter.

By requiring the fulfilment of some additional properties, we can reduce the number of possibilities for the values of a, b, c, d, e, f making feasible to present the complete families of aggregation functions of the form

$$f(x, y) = \begin{cases} 0 & \text{if } y \leq 1 - x \\ ax^2 + by^2 + cxy + dx + ey + f & \text{if } y > 1 - x, \end{cases}$$

where a, b, c, d, e, f are real numbers such that $a^2 + b^2 + c^2 > 0$. satisfying these properties.

Continuity

Proposition

The following statements are equivalent:

(i) *f is a continuous aggregation function.*

(ii) *f is given by $f(x, y) =$*

$$\begin{cases} 0 & \text{if} \\ y \leq 1 - x, & \\ ax^2 + by^2 + (a+b)xy + (1-2a-b)x + (1-a-2b)y + a + b - 1 & \text{if} \\ y > 1 - x, & \end{cases}$$

where $-1 \leq a, b \leq 1$ and $a^2 + b^2 > 0$.

Neutral element

Proposition

Let f be a binary function of the form

$$f(x,y)=\begin{cases}0 & \textit{if } y\le 1-x\\ ax^2+by^2+cxy+dx+ey+f & \textit{if } y>1-x,\end{cases}$$

where a,b,c,d,e,f are real numbers such that $a^2+b^2+c^2>0$. The following statements are equivalent:

(i) *f is an aggregation function satisfying* **(NE(e))**.

(ii) *$e=1$ and f is given by*

$$f(x,y)=\begin{cases}0 & \textit{if } y\le 1-x,\\ cxy+(1-c)x+(1-c)y+c-1 & \textit{if } y>1-x,\end{cases}$$

where $0<c\le 1$.

Symmetry

Proposition

The following statements are equivalent:

(i) *f is a continuous aggregation function satisfying* **(SYM)**.

(ii) *f is given by*

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ a(x^2+y^2) + 2axy + (1-3a)(x+y) + 2a - 1 & \text{if } y > 1-x, \end{cases}$$

where $a \in [-1, 0[\cup]0, 1]$.

Moreover, if f satisfies **(NE(e))** *then f satisfies also* **(SYM)**.

Idempotency

Proposition

If f is continuous or satisfies **(NE(e))**, *then f is not idempotent in its positive region.*

Contrarily, there exist solutions when we join **(ID)** in the positive region with **(SYM)**.

Proposition

The following statements are equivalent:

(i) *f is an aggregation function satisfying* **(SYM)** *and* **(ID)** *in its positive region.*

(ii) *f is given by*

$$f(x,y) = \begin{cases} 0 & \text{if } y \leq 1-x, \\ a(x^2+y^2) - 2axy + \frac{1}{2}(x+y) & \text{if } y > 1-x, \end{cases}$$

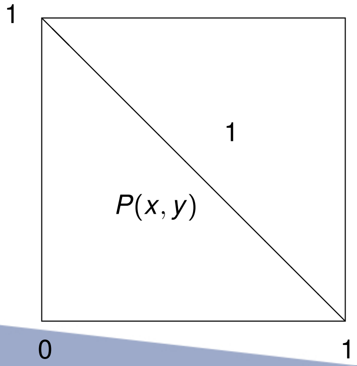
where $a \in \left[-\frac{1}{2}, 0\right[\cup \left]0, \frac{1}{2}\right]$.

Polynomial Binary Aggregation Functions with a non trivial 1-region

We will study functions of the form

$$f(x,y) = \begin{cases} P(x,y) & \text{if } y \leq 1-x, \\ 1 & \text{if } y > 1-x, \end{cases}$$

where $P(x,y)$ is a polynomial of degree 1 or 2.



Example

All the following cases are aggregation functions:

(i) A construction method of aggregation functions is based on assigning a polynomial aggregation function as $P(x, y)$ and impose the value 1 all over N_c. This construction method does not provide the whole family of aggregation functions of the previous form since for instance, it does not provide any continuous aggregation function.

(ii) The Łukasiewicz t-conorm $S_{\mathbf{L}}$ is an aggregation function of the previous form , just taking as $P(x, y)$ the polynomial $P(x, y) = x + y$. Note that this operator is continuous.

(iii) The following family of aggregation functions of the previous form proves the existence of operators of this kind for any degree n

$$f(x, y) = \begin{cases} 1 - \frac{(1-x)^n + (1-y)^n}{2} & \text{if } y < 1 - x, \\ 1 & \text{if } y \geq 1 - x. \end{cases}$$

Both forms are connected through the duality as the following result states.

Proposition

$f(x, y)$ is a binary function of the form

$$f(x,y) = \begin{cases} 0 & \textit{if } y \leq 1 - x, \\ P(x,y) & \textit{if } y > 1 - x, \end{cases}$$

if and only if $1 - f(1-x, 1-y)$ is a binary function of the form

$$\begin{cases} P(x,y) & \textit{if } y \leq 1 - x, \\ 1 & \textit{if } y > 1 - x, \end{cases}$$

As a consequence, this result allows us to rewrite easily all the results presented previously to results involving aggregation functions of this form.

For the sake of clarity, we show for instance the characterization of all aggregation functions of the previous form with degree less or equal to 1.

Theorem

Let f be a binary function

$$f(x,y) = \begin{cases} P(x,y) & \textit{if } y \leq 1-x, \\ 1 & \textit{if } y > 1-x, \end{cases}$$

with $P(x,y)$ a polynomial of degree less or equal to 1*. The following statements are equivalent:*

i) *f is a binary aggregation function.*

ii) *There exist $a, b \in [0,1]$ such that $f = f_{a,b}$ where $f_{a,b} : [0,1]^2 \to [0,1]$ is given by*

$$f_{a,b}(x,y) = \begin{cases} ax+by & \textit{if } y < 1-x, \\ 1 & \textit{if } y \geq 1-x. \end{cases}$$

Conclusions and future work

Conclusions and future work

- We have investigated aggregation functions which have a 0 (or 1)-region delimited by N_c and polynomial function as expression in the other sub-domain.
- We have characterized all binary aggregation functions with 0 (or 1)-region delimited by N_c and defined by a polynomial of degree less than or equal to 1 in the other sub-domain.
- Additional properties have been studied, such as the idempotency in the non-constant region, symmetry, continuity or the existence of a neutral element among others.
- A similar study for polynomials of degree 2 has also been developed.

Conclusions and future work

- We want to complete the study of the additional properties when an underlying polynomial of degree 2 is considered.
- Some general results for polynomials of degree n could also be investigated.
- Finally, another fuzzy negation to delimitate the constant region could be also considered.

Thank you very much!