On linear and quadratic constructions of fuzzy implication functions

S. Massanet J. V. Riera J. Torrens

SCOPIA research group
Dept. Maths and Computer Science
University of the Balearic Islands
Palma, Mallorca

Balearic Islands Health Research Institute (IdISBa)
Palma, Mallorca

http://scopia.uib.eu

Outline

1. Motivation
   - Fuzzy implication functions
   - Construction methods from other known implications
2. Construction of fuzzy implication functions from ternary functions
   - The trivial case of linear functions
3. Quadratic constructions of fuzzy implication functions
   - Preservation of (NP) and (IP)
   - Preservation of (IP) and (CP(N_c))
4. Conclusions and future work

Motivation

Fuzzy implication functions

Fuzzy implication functions have been deeply studied in last decades due to their great number of applications:

Fuzzy implication functions

Fuzzy implication functions have been deeply studied in last decades due to their great number of applications:

- approximate reasoning,

Fuzzy implication functions

Fuzzy implication functions have been deeply studied in last decades due to their great number of applications:

- approximate reasoning,
- fuzzy control,

Fuzzy implication functions

Fuzzy implication functions have been deeply studied in last decades due to their great number of applications:

- approximate reasoning,
- fuzzy control,
- image processing,

Fuzzy implication functions

Fuzzy implication functions have been deeply studied in last decades due to their great number of applications:

- approximate reasoning,
- fuzzy control,
- image processing,
- fuzzy *DI*-subsethoods…

Choice of the fuzzy implication function

For each of these applications, different fuzzy implication functions satisfying different additional properties are needed. This leads to the necessity of disposing of a bunch of fuzzy implication functions in order to pick the fuzzy implication function that satisfies those concrete additional properties needed for the corresponding application.

Choice of the fuzzy implication function

For each of these applications, different fuzzy implication functions satisfying different additional properties are needed. This leads to the necessity of disposing of a bunch of fuzzy implication functions in order to pick the fuzzy implication function that satisfies those concrete additional properties needed for the corresponding application.

Definition

A binary operator $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication function*, if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Additional properties

- The *left neutrality principle*,

$$I(1, y) = y, \quad y \in [0, 1]. \tag{\textbf{NP}}$$

- The *identity principle*,

$$I(x, x) = 1, \quad x \in [0, 1]. \tag{\textbf{IP}}$$

- The *ordering property*,

$$x \leq y \iff I(x, y) = 1, \quad x, y \in [0, 1]. \tag{\textbf{OP}}$$

- The *law of contraposition* with respect to a fuzzy negation N,

$$I(N(y), N(x)) = I(x, y), \quad x, y \in [0, 1], \tag{\textbf{CP(N)}}$$

- The *exchange principle*,

$$I(x, I(y, z)) = I(y, I(x, z)), \quad x, y, z \in [0, 1]. \tag{\textbf{EP}}$$

Construction methods from other known implications

One of the main strategies to generate new fuzzy implication functions is based on applying some transformations to one or two given fuzzy implication functions.

Construction methods from other known implications

One of the main strategies to generate new fuzzy implication functions is based on applying some transformations to one or two given fuzzy implication functions.

1. The N-reciprocation of a fuzzy implication function I

$$I_N(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1].$$

Construction methods from other known implications

One of the main strategies to generate new fuzzy implication functions is based on applying some transformations to one or two given fuzzy implication functions.

1. The N-reciprocation of a fuzzy implication function I

$$I_N(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1].$$

2. The φ-conjugation of a fuzzy implication function I

$$I_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \quad x, y \in [0, 1].$$

Construction methods from other known implications

One of the main strategies to generate new fuzzy implication functions is based on applying some transformations to one or two given fuzzy implication functions.

1. The N-reciprocation of a fuzzy implication function I

$$I_N(x, y) = I(N(y), N(x)), \quad x, y \in [0, 1].$$

2. The φ-conjugation of a fuzzy implication function I

$$I_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \quad x, y \in [0, 1].$$

3. The min and max operations from two given ones I, J:

$$(I \vee J)(x, y) = \max\{I(x, y), J(x, y)\}, \quad x, y \in [0, 1],$$
$$(I \wedge J)(x, y) = \min\{I(x, y), J(x, y)\}, \quad x, y \in [0, 1].$$

Construction methods from other known implications

One of the main strategies to generate new fuzzy implication functions is based on applying some transformations to one or two given fuzzy implication functions.

1. The N-reciprocation of a fuzzy implication function I
$$I_N(x,y) = I(N(y),N(x)), \quad x,y \in [0,1].$$
2. The φ-conjugation of a fuzzy implication function I
$$I_\varphi(x,y) = \varphi^{-1}(I(\varphi(x),\varphi(y))), \;\; x,y \in [0,1].$$
3. The $\min$ and $\max$ operations from two given ones I, J:
$$(I \vee J)(x,y) = \max\{I(x,y),J(x,y)\}, \;\; x,y \in [0,1],$$
$$(I \wedge J)(x,y) = \min\{I(x,y),J(x,y)\}, \;\; x,y \in [0,1].$$
4. The convex combinations of fuzzy implication functions from two given ones I, J and $\lambda \in [0,1]$:
$$I_{I,J}^{\lambda}(x,y) = \lambda \cdot I(x,y) + (1-\lambda)\cdot J(x,y), \;\; x,y \in [0,1].$$

Construction methods from other known implications

5. Horizontal and vertical threshold generation methods from two fuzzy implication functions.

Construction methods from other known implications

5. Horizontal and vertical threshold generation methods from two fuzzy implication functions.
6. The *FNI*-method of a fuzzy implication function I

$$I_{FNI}(x, y) = F(N(x), I(x, y)), \quad x, y \in [0, 1].$$

Construction methods from other known implications

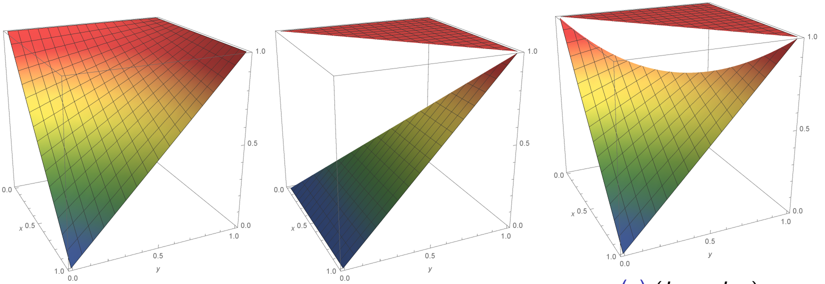
5. Horizontal and vertical threshold generation methods from two fuzzy implication functions.
6. The *FNI*-method of a fuzzy implication function I

$$I_{FNI}(x, y) = F(N(x), I(x, y)), \quad x, y \in [0, 1].$$

7. The $\circledast$-composition of two fuzzy implication functions I and J:

$$(I \circledast J)(x, y) = I(x, J(x, y)), \quad x, y \in [0, 1].$$

Construction methods from other known implications



(a) $I_{\mathbf{RC}}$ (b) $I_{\mathbf{GD}}$ (c) $(I_{\mathbf{RC}} \circledast I_{\mathbf{GD}})$

Construction methods from other known implications

There are two main important points in these construction methods:

Construction methods from other known implications

There are two main important points in these construction methods:

- They should preserve as many properties as possible.
- The new obtained fuzzy implication function has an expression easy to compute and to implement.

Goal

To propose a very easy construction method (based on quadratic functions of three variables) which preserves many properties, including **(CP(N_c))**.

Construction of fuzzy implication functions from ternary functions

Constructions from ternary functions

The construction is based on the function $I_F : [0,1]^2 \to \mathbb{R}$ given by

$$I_F(x,y) = F(x,y,I(x,y)) \qquad \text{for all } x,y \in [0,1],$$

where

- $F : \mathbb{R}^3 \to \mathbb{R}$ is a ternary function,
- I is a fuzzy implication function.

Constructions from ternary functions

The construction is based on the function $I_F : [0,1]^2 \to \mathbb{R}$ given by

$$I_F(x,y) = F(x,y,I(x,y)) \qquad \text{for all } x,y \in [0,1],$$

where

- $F : \mathbb{R}^3 \to \mathbb{R}$ is a ternary function,
- I is a fuzzy implication function.

Remark

The process is the same that was used by Kolesárová and Mesiar in

A. Kolesárová and R. Mesiar. On linear and quadratic constructions of aggregations functions. Fuzzy Sets and Systems, 268:1–14, 2015.

to construct new aggregation functions from old ones and, specially, new semi-copulas and quasi-copulas.

Some observations

- x, y and $I(x, y)$ are in $[0, 1]$, but not necessarily, $F(x, y, I(x, y))$!

Some observations

- x, y and $I(x, y)$ are in $[0, 1]$, but not necessarily, $F(x, y, I(x, y))$!
- We are looking for functions F such that $F(x, y, I(x, y))$ is a fuzzy implication function for any fuzzy implication function I.

Some observations

- x, y and $I(x, y)$ are in $[0, 1]$, but not necessarily, $F(x, y, I(x, y))$!
- We are looking for functions F such that $F(x, y, I(x, y))$ is a fuzzy implication function for any fuzzy implication function I.
- $F(x, y, z) = z$ is not an interesting solution since in this case, $I_F(x, y) = F(x, y, I(x, y)) = I(x, y)$.

Some observations

- x, y and $I(x, y)$ are in $[0, 1]$, but not necessarily, $F(x, y, I(x, y))$!
- We are looking for functions F such that $F(x, y, I(x, y))$ is a fuzzy implication function for any fuzzy implication function I.
- $F(x, y, z) = z$ is not an interesting solution since in this case, $I_F(x, y) = F(x, y, I(x, y)) = I(x, y)$.
- We will study the particular case of polynomial ternary functions.

An interesting example

Example

Consider the family of functions $F_n(x, y, z) = z^n$ *for all* $n \geq 1$. *This family leads to the construction methods given by*

$$I_{F_n}(x, y) = I(x, y)^n \qquad \text{for all } x, y \in [0, 1].$$

All these constructions preserve **(IP)**, **(OP)** *and* **(CP(N))**.

An interesting example

Example

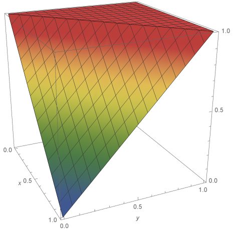
Consider the family of functions $F_n(x, y, z) = z^n$ *for all* $n \geq 1$. *This family leads to the construction methods given by*

$$I_{F_n}(x, y) = I(x, y)^n \qquad \text{for all } x, y \in [0, 1].$$

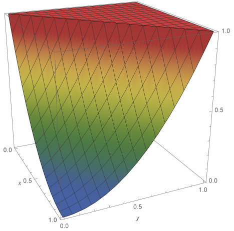
All these constructions preserve **(IP)**, **(OP)** *and* **(CP(N))**.

Moreover, if we consider the sequence $(I_{F_n})_n$, *this converges to the least fuzzy implication function with the same 1-region than* I.

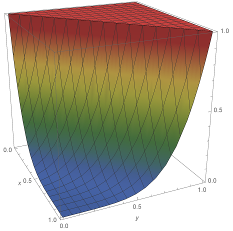
An interesting example



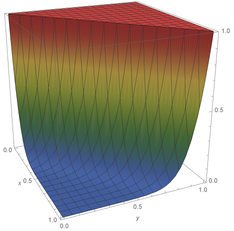
(a) $I_{\mathbf{L}}$



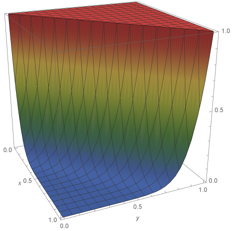
(b) I_{F_2}



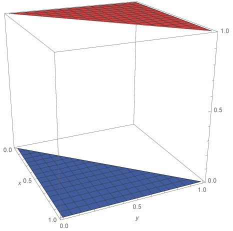
(c) I_{F_5}



(d) $I_{F_{10}}$



(e) $I_{F_{50}}$



(f) I_{F_∞}

Linear functions

Let us begin our study with the most simple polynomial functions: linear functions, i.e., functions $F : \mathbb{R}^3 \to \mathbb{R}$ such that

$$F(x, y, z) = ax + by + cz + d$$

with $a, b, c, d \in \mathbb{R}$.

Linear functions

Let us begin our study with the most simple polynomial functions: linear functions, i.e., functions $F:\mathbb{R}^3 \to \mathbb{R}$ such that

$$F(x, y, z) = ax + by + cz + d$$

with $a, b, c, d \in \mathbb{R}$.

Proposition

Let $F:\mathbb{R}^3 \to \mathbb{R}$ be a linear function. Then the following statements are equivalent:

i) *For each I in $\mathcal{I}$ the function I_F is also in $\mathcal{I}$.*

ii) *The function F is given by the projection $F(x, y, z) = z$, that is, $a = b = d = 0$ and $c = 1$, and therefore I_F is the proper I.*

Quadratic constructions of fuzzy implication functions

Quadratic functions

Let us consider functions $F : [0,1]^3 \to [0,1]$ given by

$$F(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j,$$

with $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$.

Quadratic functions

Let us consider functions $F : [0,1]^3 \to [0,1]$ given by

$$F(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j,$$

with $a,b,c,d,e,f,g,h,i,j \in \mathbb{R}$.

Proposition

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function and $I : [0,1]^2 \to [0,1]$ a fuzzy implication function. Then the following statements are equivalent:

i) *I_F fulfils boundary conditions* **(I3)** *and $I_F(0,x) = I_F(x,1) = 1$ for all $x \in [0,1]$.*

ii) $a = b = 0,\ e = -(d+g), h = -f,\ j = -g,$ *and* $i = 1 - c + g.$

Quadratic functions

Let us consider functions $F : [0,1]^3 \to [0,1]$ given by

$$F(x,y,z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j,$$

with $a, b, c, d, e, f, g, h, i, j \in \mathbb{R}$.

Proposition

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function and $I : [0,1]^2 \to [0,1]$ a fuzzy implication function. Then the following statements are equivalent:

i) *I_F fulfils boundary conditions* **(I3)** *and $I_F(0,x) = I_F(x,1) = 1$ for all $x \in [0,1]$.*

ii) $a = b = 0,\ e = -(d+g), h = -f,\ j = -g,$ *and* $i = 1 - c + g.$

This leads to

$$F(x,y,z) = cz^2 + dxy - (d+g)xz + fyz + gx - fy + (1-c+g)z - g$$

with $c, d, g, f \in \mathbb{R}$.

Quadratic functions

However, to ensure monotonicities, the four parameters must satisfy some complex additional properties quite difficult to manage in practice...
Therefore, we will impose the preservation of some additional properties to look for manageable solutions.

Quadratic functions

However, to ensure monotonicities, the four parameters must satisfy some complex additional properties quite difficult to manage in practice...
Therefore, we will impose the preservation of some additional properties to look for manageable solutions.

Proposition

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function and $I : [0,1]^2 \to [0,1]$ a fuzzy implication function. Then the next statements hold:

i) *If I satisfies* **(NP)**, *then I_F fulfils* **(NP)** *if and only if $f + c = 0$.*

ii) *If I satisfies* **(IP)**, *then I_F fulfils* **(IP)** *if and only if $d = 0$.*

iii) *If I satisfies* **(CP(N_c))**, *then I_F fulfils* **(CP(N_c))** *if and only if $d + g - f = 0$.*

Quadratic functions

However, to ensure monotonicities, the four parameters must satisfy some complex additional properties quite difficult to manage in practice... Therefore, we will impose the preservation of some additional properties to look for manageable solutions.

Proposition

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function and $I : [0,1]^2 \to [0,1]$ a fuzzy implication function. Then the next statements hold:

i) *If I satisfies* **(NP)**, *then I_F fulfils* **(NP)** *if and only if $f + c = 0$.*

ii) *If I satisfies* **(IP)**, *then I_F fulfils* **(IP)** *if and only if $d = 0$.*

iii) *If I satisfies* **(CP(N_c))**, *then I_F fulfils* **(CP(N_c))** *if and only if $d + g - f = 0$.*

However, I_F may still not be a fuzzy implication function!

Quadratic constructions preserving (NP) and (IP)

Theorem

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function. Then the following statements are equivalent:

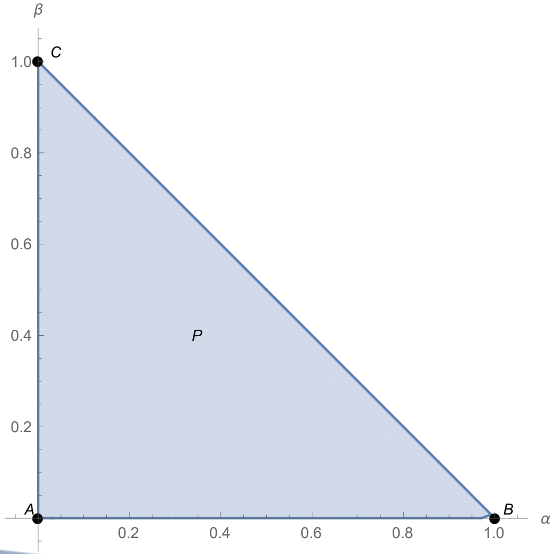
i) *For each I in $\mathcal{I}_{\mathbf{IP},\mathbf{NP}}$, the function I_F is also in $\mathcal{I}_{\mathbf{IP},\mathbf{NP}}$.*

ii) *$I_F = I_{\alpha,\beta}$, where $I_{\alpha,\beta}$ is given by*

$$I_{\alpha,\beta}(x,y) = \alpha I(x,y)^2 + \beta x I(x,y) - \alpha y I(x,y) - \beta x + \alpha y + (1-\alpha-\beta) I(x,y) + \beta$$

with α, β fulfilling the conditions $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1-\alpha$ and $I \in \mathcal{I}$.

Quadratic constructions preserving (NP) and (IP)



Quadratic constructions preserving (IP) and (CP(N_c))

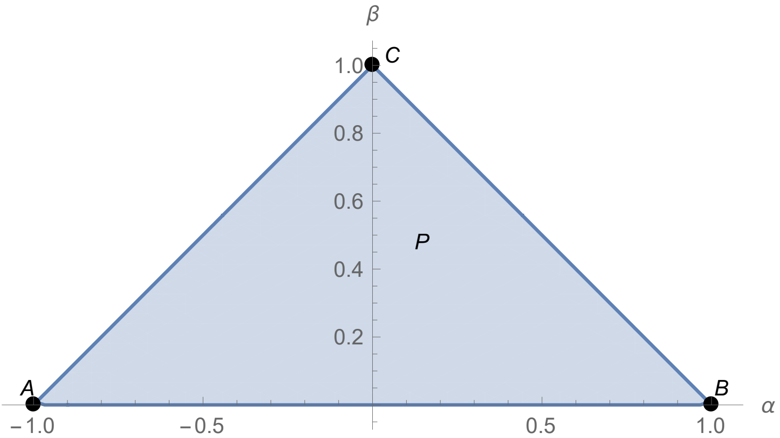
Theorem

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a quadratic function. For each I in $\mathcal{I}_{\mathbf{IP},\mathbf{CP}(\mathbf{N_c})}$, the function I_F is also in $\mathcal{I}_{\mathbf{IP},\mathbf{CP}(\mathbf{N_c})}$, if $I_F = I_{\alpha,\beta}$, where

$$I_{\alpha,\beta}(x,y) = \alpha I(x,y)^2 + \beta x I(x,y) - \beta y I(x,y) - \beta x + \beta y + (1-\alpha-\beta) I(x,y) + \beta \quad (1)$$

with α, β such that either $-1 \leq \alpha \leq 0$ and $0 \leq \beta \leq \alpha + 1$ or $0 < \alpha \leq 1$ and $0 \leq \beta \leq -\alpha + 1$.

Quadratic constructions preserving (IP) and (CP(N_c))



Preservation of (IP) and (CP(N_c))

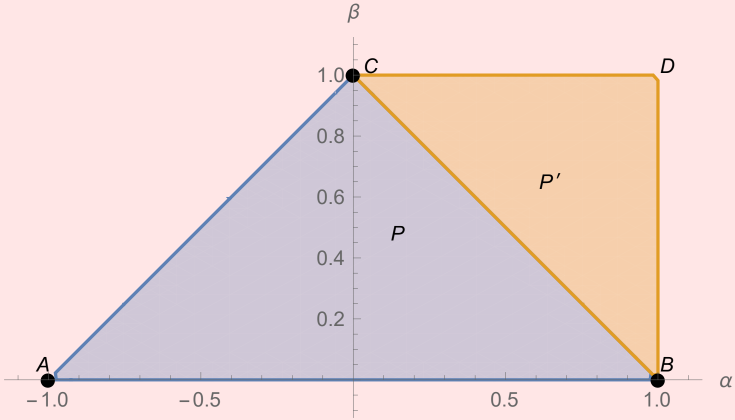
Open problem

This is not a complete characterization since there could be more eligible parameter values. Indeed, we have the feeling that the previous region can be expanded to

Preservation of (IP) and (CP(N_c))

Open problem

This is not a complete characterization since there could be more eligible parameter values. Indeed, we have the feeling that the previous region can be expanded to



Conclusions and future work

Conclusions

In this paper,

- we have studied some novel construction methods of fuzzy implication functions from a given one based on ternary polynomial functions,

Conclusions

In this paper,

- we have studied some novel construction methods of fuzzy implication functions from a given one based on ternary polynomial functions,
- we have proved that the quadratic method yields infinitely many different novel fuzzy implication functions,

Conclusions

In this paper,

- we have studied some novel construction methods of fuzzy implication functions from a given one based on ternary polynomial functions,
- we have proved that the quadratic method yields infinitely many different novel fuzzy implication functions,
- we have studied the preservation of **(NP)** and **(IP)**, or **(IP)** and **(CP(N_c))** by this method in great detail.

Future work

As a future work,

- we want to completely solve the preservation of **(IP)** and **(CP(N_c))** by this method,

Future work

As a future work,

- we want to completely solve the preservation of **(IP)** and **(CP(N_c))** by this method,
- we want to study the case of the preservation of **(CP(N_c))** and **(NP)** by this method (already promising results!),

Future work

As a future work,

- we want to completely solve the preservation of **(IP)** and **(CP(N_c))** by this method,
- we want to study the case of the preservation of **(CP(N_c))** and **(NP)** by this method (already promising results!),
- many other combinations of two or more additional properties could be considered. . .


Thank you for your attention!