# Aggregation Functions Given by Polynomial Functions

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Soft Computing, Image Processing and Aggregation

# Main points of this talk

1. Motivation
   - Aggregation functions
2. Preliminaries
   - Aggregation functions: basic properties
3. Polynomial binary aggregation functions
   - Degree 1
   - Degree 2
4. Polynomial n-ary aggregation functions
   - General case
   - Degree 1 and 2
5. Conclusions and future work

# Motivation

# Motivation

- Aggregation functions play a fundamental role in social and scientific sciences because in all of them it becomes necessary at some point to merge several input data into a representative output value.
- This is the main reason for which the theoretical study of aggregation functions have experienced an important growth in last decades.

# Motivation

- Taking in mind the applications (for instance, decision making process, image processing, approximate reasoning, ...), one can look for aggregation functions having expressions as simple as possible.
- A first step could be to study aggregation functions whose expression is given by polynomial or rational functions of different degrees.

# Motivation

This was already done in some particular cases.

- All the rational Archimedean continuous t-norms are characterized as the well-known Hamacher class which contains the t-norms given by the following expression

$$T_\alpha(x,y) = \frac{xy}{\alpha + (1-\alpha)(x+y-xy)}, \quad x,y \in [0,1]$$

  with $\alpha \geq 0$

- All the rational uninorms were characterized as those whose expression is given by

$$U_e(x,y) = \frac{(1-e)xy}{(1-e)xy + e(1-x)(1-y)}$$

  if $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and, $U(1,0) = U(0,1) = 0$ or $U(0,1) = U(1,0) = 1$.

# Motivation and Main Goal

- A similar study for polynomial and rational fuzzy implication functions have been proposed recently.
- Now, we want to investigate aggregation functions whose expressions are given by polynomial functions.

# Preliminaries

## Definition

An *n-ary aggregation function* is a function of $n > 1$ arguments that maps the ($n$-dimensional) unit cube onto the unit interval $f : [0,1]^n \to [0,1]$, with the properties

i) $f(\underbrace{0, 0, \ldots, 0}_{n-\text{times}}) = 0$ and $f(\underbrace{1, 1, \ldots, 1}_{n-\text{times}}) = 1$.

ii) $\mathbf{x} \leq \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$.

In particular, a 2-ary aggregation function will be called a *binary aggregation function*.

# Properties of n-ary aggregation functions

- The *idempotency*,

$$f(\mathbf{x}) = \mathbf{x}, \quad \text{for all } \mathbf{x} = (x, \ldots, x), x \in [0, 1]. \qquad \textbf{(ID)}$$

- The *symmetry*,

$$f(x_1, x_2, \ldots, x_n) = f(x_{P(1)}, x_{P(2)}, \ldots, x_{P(n)}), \qquad \textbf{(SYM)}$$

  for all $\mathbf{x} = (x_1, \ldots, x_n)$ and permutation $P = (P(1), \ldots, P(n))$ of $(1, 2, \ldots, n)$.

# Properties of n-ary aggregation functions

- The *zero divisor property* with $a \in (0, 1)$,

$$f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = 0, \qquad (\mathbf{ZD}(\mathbf{a}))$$

for some $\mathbf{x} \in [0, 1]^n$ with $x_i = a$ with $a$ in any position and $x_i > 0$ for all $1 \leq x_i \leq n$.

- The *one divisor property* with $a \in (0, 1)$,

$$f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = 1, \qquad (\mathbf{OD}(\mathbf{a}))$$

for some $\mathbf{x} \in [0, 1]^n$ with $x_i = a$ with $a$ in any position and $x_i < 1$ for all $1 \leq x_i \leq n$.

# Additional properties for binary aggregation functions

- The *associativity*,

$$f(f(x,y),z) = f(x,f(y,z)), \quad \text{for all } x,y,z \in [0,1]. \qquad (\mathbf{ASS})$$

- The *left neutral element property* with a fixed $e \in [0,1]$,

$$f(e,y) = y, \quad \text{for all } y \in [0,1]. \qquad (\mathbf{L-NE(e)})$$

- The *right neutral element property* with a fixed $e \in [0,1]$,

$$f(x,e) = x, \quad \text{for all } x \in [0,1]. \qquad (\mathbf{R-NE(e)})$$

- The *neutral element property* with a fixed $e \in [0,1]$,

$$f(e,x) = f(x,e) = x, \quad \text{for all } x \in [0,1]. \qquad (\mathbf{NE(e)})$$

# Additional properties for binary aggregation functions

- The *left absorbing element property* with a fixed $a \in [0,1]$,

$$f(a,y) = a, \quad \text{for all } y \in [0,1]. \tag{$\mathbf{L-AE(a)}$}$$

- The *right absorbing element property* with a fixed $a \in [0,1]$,

$$f(x,a) = a, \quad \text{for all } x \in [0,1]. \tag{$\mathbf{R-AE(a)}$}$$

- The *absorbing element property* with a fixed $a \in [0,1]$,

$$f(a,x) = f(x,a) = a, \quad \text{for all } x \in [0,1]. \tag{$\mathbf{AE(a)}$}$$

# Polynomial binary aggregation functions

## Definition

Let $n \in \mathbb{N}$. A binary operator $f : [0,1]^2 \to [0,1]$ is called a *polynomial binary aggregation function of degree n* if it is a binary aggregation function and its expression is given by

$$f(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$$

for all $x, y \in [0,1]$ where $a_{ij} \in \mathbb{R}$ and there exist some $0 \le i, j \le n$ with $i + j = n$ such that $a_{ij} \neq 0$.

# Polynomial aggregation function of degree n

## Example

Let us consider the parametrized family of polynomial fuzzy implication functions of degree $n$

$$I_n(x, y) = 1 - x^{n-1} + x^{n-1}y$$

for all $x, y \in [0, 1]$ and $n \geq 2$. Given a fuzzy implication function $I$, $D(x, y) = I(1 - x, y)$ is a disjunctor. From this result,

$$A_n(x, y) = I_n(1 - x, y) = 1 - (1 - x)^{n-1} + (1 - x)^{n-1}y$$

for all $x, y \in [0, 1]$ and $n \geq 2$ is a family of disjunctors. Consequently, there exist polynomial binary aggregation functions of any degree $n \in \mathbb{N}$ with $n \geq 2$.

### Remark

Note that some well-known aggregation functions whose expression is piecewise polynomial are not considered as polynomial aggregation functions in this approach. For instance,

$$T_{\mathbf{M}}(x,y) = \min\{x,y\}, \qquad T_{\mathbf{L}}(x,y) = \max\{0, x+y-1\},$$
$$S_{\mathbf{M}}(x,y) = \max\{x,y\}, \qquad S_{\mathbf{L}}(x,y) = \min\{1, x+y\},$$

as well as any OWA operators are not polynomial since they do not satisfy the expression given previously.

# Immediate properties

## Proposition

- *All polynomial binary aggregation functions are continuous.*
- *There is no polynomial binary aggregation function with either zero or one divisors.*

# Necessary and sufficient conditions to be a PBAF

**Theorem**

*A polynomial* $p(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *of degree n is a polynomial binary aggregation function if, and only if, the following properties hold:*

(i) $p(0,0) = 0$ *and* $p(1,1) = 1$.

(ii) $\frac{\partial p(x,y)}{\partial x} \ge 0$ *for all* $x, y \in [0,1]$.

(iii) $\frac{\partial p(x,y)}{\partial y} \ge 0$ *for all* $x, y \in [0,1]$.

From first property of previous Theorem,

## Proposition

*Let* $p(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$ *be a polynomial of degree n. Then we have the following equivalences:*

(i) $p(0,0) = 0$ *if, and only if,* $a_{00} = 0$*.*

(ii) $p(1,1) = 1$ *if, and only if,* $\sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} = 1$*.*

# Degree 1

# Degree 1

## Theorem

*Let $p(x, y)$ be a polynomial of degree 1. The following statements are equivalent:*

i) *$p$ is a polynomial binary aggregation function of degree 1.*

ii) *$p$ belongs to the family of binary weighted arithmetic means, given by*

$$p(x, y) = \alpha x + (1 - \alpha) y$$

*for all $x, y \in [0, 1]$ and $\alpha \in [0, 1]$.*

## Degree 2

# Degree 2

## Goal: To characterize all PBAF of degree 2

Now we deal with the characterization of all polynomial binary aggregation functions of degree 2, i.e., those whose expression is given by

$$f(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$$

with $a_{11}^2 + a_{20}^2 + a_{02}^2 \neq 0$.

## Theorem

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

i) *$p$ is a polynomial binary aggregation function of degree 2.*

ii) *$p$ is given by the following expression*

$$\begin{aligned} p(x,y) &= a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 \\ &+ (1 - a_{10} - a_{01} - a_{11} - a_{20})y^2 \end{aligned}$$

*for all $x, y \in [0, 1]$ where the coefficients satisfy the following conditions:*

1. $a_{11}^2 + a_{20}^2 + a_{02}^2 \neq 0$.
2. *Either*
   - 2.1 $a_{10} = 0$ *and* $a_{11}, a_{20} \geq 0$*, or*
   - 2.2 $a_{10} > 0$ *and* $a_{11} = -a_{10}$ *and* $a_{20} \geq 0$*, or*
   - 2.3 $a_{10} > 0$ *and* $-a_{10} < a_{11} \leq 0$ *and* $a_{20} \geq \frac{1}{2}(-a_{10} - a_{11})$*, or*
   - 2.4 $a_{10} > 0$ *and* $a_{11} > 0$ *and* $a_{20} \geq -\frac{a_{10}}{2}$.
3. *Either*
   - 3.1 $a_{01} = 0$ *and* $a_{11} \geq 0$ *and* $a_{20} \leq 1 - a_{10} - a_{11}$*, or*
   - 3.2 $a_{01} > 0$ *and* $-a_{01} \leq a_{11} \leq 0$ *and* $a_{20} \leq \frac{1}{2}(2 - a_{01} - 2a_{10} - a_{11})$*, or*
   - 3.3 $a_{01} > 0$ *and* $a_{11} > 0$ *and* $a_{20} \leq \frac{1}{2}(2 - a_{01} - 2a_{10} - 2a_{11})$.

The conditions on the coefficients given in the previous result could be merged among them to clarify which coefficients values are suitable to define a polynomial binary aggregation function

### Example

(i) Joining properties 1, 2.1 and 3.1, we obtain the product t-norm $T_{\mathbf{P}}(x, y) = xy$ and the family

$$f(x, y) = \alpha xy + \beta x^2 + (1 - \alpha - \beta)y^2$$

with $0 \leq \alpha < 1$ and $0 \leq \beta \leq 1 - \alpha$.

(ii) Joining properties 1, 2.1 and 3.2, we obtain the function $f(x, y) = 2y - y^2$ and the two following families:
- $f(x, y) = \alpha y + \beta x^2 + (1 - \alpha - \beta)y^2$ with $\alpha \in (0, 2) \setminus \{1\}$ and $0 \leq \beta \leq \frac{2-\alpha}{2}$.
- $f(x, y) = y + \alpha x^2 - \alpha y^2$ with $0 < \alpha \leq \frac{1}{2}$.

**Let us characterize polynomial binary aggregation functions of degree 2 fulfilling some additional properties: symmetry, idempotency, (left/right) neutral element and (left/right) absorbing element**

# Symmetry

## Proposition

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

i) *$p$ is a polynomial binary aggregation function of degree 2 satisfying (SYM).*

ii) *$p$ is given by the following expression*

$$p(x, y) = \alpha(x + y) + \beta xy + \frac{1}{2}(1 - 2\alpha - \beta)(x^2 + y^2)$$

*for all $x, y \in [0, 1]$ where the coefficients satisfy $0 \leq \alpha \leq 1$ and $-\alpha \leq \beta \leq 1 - \alpha$ excluding $\beta = 0$ when $\alpha = \frac{1}{2}$.*

# Idempotency

## Proposition

*Let $p(x, y)$ be a polynomial aggregation function of degree 2. The following statements are equivalent:*

i) *$p$ satisfies (ID).*

ii) *$p$ is given by the following expression*

$$p(x, y) = \alpha x + (1 - \alpha)y + \beta xy + \gamma x^2 + (-\beta - \gamma)y^2$$

*for all $x, y \in [0, 1]$ where the coefficients satisfy*

- $\alpha = 0$ *and either*
  - (i) $0 \leq \beta < \frac{1}{2}$ *and* $0 \leq \gamma < \frac{1}{2}(1 - 2\gamma)$
  - (ii) $\beta = \frac{1}{2}$ *and* $\gamma = -\frac{1}{2}$
- $\alpha = 1$ *and either*
  - (i) $\beta = 0$ *and* $-\frac{1}{2} \leq \gamma < 0$
  - (ii) $0 < \beta < \frac{1}{2}$ *and* $-\frac{1}{2} < \gamma \leq -\beta$
  - (iii) $\beta = \frac{1}{2}$ *and* $\gamma = -\frac{1}{2}$

# Idempotency

## Proposition

*Let $p(x, y)$ be a polynomial aggregation function of degree 2. The following statements are equivalent:*

i) *$p$ satisfies (ID).*

ii) *$p$ is given by the following expression*

$$p(x, y) = \alpha x + (1 - \alpha)y + \beta xy + \gamma x^2 + (-\beta - \gamma)y^2$$

*for all $x, y \in [0, 1]$ where the coefficients satisfy*

- $0 < \alpha \leq \frac{1}{2}$ *and either*
  - (i) $\beta = -\alpha$ *and* $0 \leq \gamma < \frac{1}{2}(1 - \alpha - \beta)$
  - (ii) $-\alpha \leq \beta \leq 0$ *and* $\frac{1}{2}(-\alpha - \beta) \leq \gamma \leq \frac{1}{2}(1 - \alpha - \beta)$
  - (iii) $0 < \beta < \frac{1-\alpha}{2}$ *and* $-\frac{\alpha}{2} \leq \gamma < \frac{1}{2}(1 - \alpha - 2\beta)$
  - (iv) $\beta = \frac{1-\alpha}{2}$ *and* $-\frac{\alpha}{2} < \gamma \leq 0$
  - (v) $\frac{1-\alpha}{2} < \beta < \frac{1}{2}$ *and* $-\frac{\alpha}{2} \leq \gamma \leq \frac{1}{2}(1 - \alpha - 2\beta)$
  - (vi) $\beta = \frac{1}{2}$ *and* $\gamma = -\frac{\alpha}{2}$

# Idempotency

## Proposition

*Let $p(x, y)$ be a polynomial aggregation function of degree 2. The following statements are equivalent:*

i) *$p$ satisfies (ID).*

ii) *$p$ is given by the following expression*

$$p(x, y) = \alpha x + (1 - \alpha)y + \beta xy + \gamma x^2 + (-\beta - \gamma)y^2$$

*for all $x, y \in [0, 1]$ where the coefficients satisfy*

- $\frac{1}{2} < \alpha < 1$ *and either*
  - (i) $-1 + \alpha \leq \beta \leq 0$ *and* $\frac{1}{2}(-\alpha - \beta) \leq \gamma \leq \frac{1}{2}(1 - \alpha - \beta)$
  - (ii) $0 < \beta < \frac{1-\alpha}{2}$ *and* $-\frac{\alpha}{2} \leq \gamma < \frac{1}{2}(1 - \alpha - 2\beta)$
  - (iii) $\beta = \frac{1-\alpha}{2}$ *and* $-\frac{\alpha}{2} \leq \gamma \leq 0$
  - (iv) $\frac{1-\alpha}{2} < \beta < \frac{1}{2}$ *and* $-\frac{\alpha}{2} \leq \gamma \leq \frac{1}{2}(1 - \alpha - 2\beta)$
  - (v) $\beta = \frac{1}{2}$ *and* $\gamma = -\frac{\alpha}{2}$

In particular, the following result characterizes the idempotent and symmetric polynomial binary aggregation functions of degree 2.

## Corollary

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

i) *$p$ is a polynomial binary aggregation function of degree 2 satisfying (SYM) and (ID).*

ii) *$p$ is given by the following expression*

$$p(x, y) = \frac{1}{2}(x + y) + \beta xy - \frac{\beta}{2}(x^2 + y^2)$$

*for all $x, y \in [0, 1]$ where $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}$.*

# Left-Right neutral element

## Proposition

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *$p$ is a polynomial binary aggregation function with left neutral element $e \in [0, 1]$.*

(ii) *One of these two cases hold:*

- *$e = 1$ and $p(x, y) = \alpha y + (1 - \alpha)xy$ with $0 \leq \alpha < 1$.*
- *$e = 0$ and $p(x, y) = \alpha x + y - \alpha xy$ with $0 < \alpha \leq 1$.*

## Proposition

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *$p$ is a polynomial binary aggregation function with right neutral element $e \in [0, 1]$.*

(ii) *One of these two cases hold:*

- *$e = 1$ and $p(x, y) = \alpha x + (1 - \alpha)xy$ with $0 \leq \alpha < 1$.*
- *$e = 0$ and $p(x, y) = x + \alpha y - \alpha xy$ with $0 < \alpha \leq 1$.*

Joining both results, the following corollary holds.

## Corollary

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *$p$ is a polynomial binary aggregation function with neutral element $e \in [0, 1]$.*

(ii) *One of these two cases hold:*

- $e = 1$ *and* $p(x, y) = xy$.
- $e = 0$ *and* $p(x, y) = x + y - xy$.

Let us study at this point the existence of a left absorbing element for these functions.

## Proposition

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *$p$ is a polynomial binary aggregation function with left absorbing element $a \in [0, 1]$.*

(ii) *One of these cases hold:*

- $a \in \{0, 1\}$ *and* $p(x, y) = \alpha x + (1 - \alpha)x^2$ *with* $\alpha \in [0, 2] \setminus \{1\}$.
- $a = 0$ *and* $p(x, y) = \alpha x + \beta xy + (1 - \alpha - \beta)x^2$ *with* $0 \leq \alpha < 2$ *and* $0 < \beta \leq \frac{2-\alpha}{2}$.
- $a = 1$ *and* $p(x, y) = \alpha x + \beta y - \beta xy + (1 - \alpha)x^2$ *with* $0 < \alpha \leq 1$ *and* $0 < \beta \leq \alpha$.
- $a = 1$ *and* $p(x, y) = \alpha x + \beta y - \beta xy + (1 - \alpha)x^2$ *with* $1 < \alpha < 2$ *and* $0 < \beta \leq 2 - \alpha$.

Let us study at this point the existence of a right absorbing element for these functions.

## Proposition

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *$p$ is a polynomial binary aggregation function with right absorbing element $a \in [0, 1]$.*

(ii) *One of these cases hold:*

- $a \in \{0, 1\}$ *and* $p(x, y) = \alpha y + (1 - \alpha)y^2$ *with* $\alpha \in [0, 2] \setminus \{1\}$.
- $a = 0$ *and* $p(x, y) = \alpha y + \beta xy + (1 - \alpha - \beta)y^2$ *with* $0 \leq \alpha < 2$ *and* $0 < \beta \leq \frac{2-\alpha}{2}$.
- $a = 1$ *and* $p(x, y) = \alpha x + \beta y - \alpha xy + (1 - \beta)y^2$ *where* $0 < \alpha < 1$ *and* $\alpha \leq \beta \leq 2 - \alpha$.
- $a = 1$ *and* $p(x, y) = x + y - xy$.

Joining both results, the following corollary holds.

## Corollary

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *$p$ is a polynomial binary aggregation function with absorbing element $a \in [0, 1]$.*

(ii) *One of these two cases hold:*

- *$a = 0$ and $p(x, y) = xy$.*
- *$a = 1$ and $p(x, y) = x + y - xy$.*

As the previous result highlights, the product t-norm and the probabilistic sum t-conorm are quite distinctive polynomial binary aggregation functions of degree 2 since they are the unique conjunctor and disjunctor, respectively. The following result makes them even more special.

## Proposition

*Let $p(x, y)$ be a polynomial of degree 2. The following statements are equivalent:*

(i) *p is an associative polynomial binary aggregation function.*

(ii) *p is either the product t-norm or the probabilistic sum t-conorm.*

# Polynomial n-ary aggregation functions

## Definition

Let $n, k \in \mathbb{N}$. An $n$-ary operator $f : [0,1]^n \to [0,1]$ is called a *polynomial aggregation function of degree $k$* if it is an $n$-ary aggregation function and its expression is given by

$$f(x_1, \ldots, x_n) = \sum_{\substack{\sum_{j=1}^{n} i_j \leq k \\ 0 \leq i_j \leq k}} a_{i_1 \ldots i_n} \prod_{j=1}^{n} x_j^{i_j}$$

for all $x, y \in [0,1]$ where $a_{i_1 \ldots i_n} \in \mathbb{R}$ and there exist some $0 \leq i_1, \ldots, i_n \leq k$ with $\sum_{j=1}^{n} i_j = k$ such that $a_{i_1 \ldots i_n} \neq 0$.

## Remark

It is clear that, given an associative polynomial binary aggregation function, a polynomial aggregation function with any number of inputs can be constructed in the usual way. However, since there are only two associative polynomial binary aggregation functions, the study of $n$-ary polynomial aggregation functions is worthy in itself.

### Theorem

*An n-ary polynomial p of degree k is a polynomial aggregation function if, and only if, the following properties hold:*

(i) $p(0, \ldots, 0) = 0$ *and* $p(1, \ldots, 1) = 1$.

(ii) $\frac{\partial p(x_1, \ldots, x_n)}{\partial x_j} \geq 0$ *for all* $x_1, \ldots, x_n \in [0, 1]$.

# Degree1

## Theorem

*Let $p$ be an $n$-ary polynomial of degree $1$. The following statements are equivalent:*

i) *$p$ is a polynomial aggregation function of degree $1$.*

ii) *$p$ belongs to the family of weighted arithmetic means.*

# Degree 2: Symmetry

## Proposition

*Let p be an n-ary polynomial of degree 2. The following statements are equivalent:*

i) *p is a polynomial aggregation function of degree 2 satisfying (SYM).*

ii) *p is given by the following expression*

$$\begin{aligned} p(x_1,\ldots,x_n) &= \alpha \sum_{j=1}^{n} x_j + \beta \sum_{1\leq j<j'\leq n} x_j x_{j'} \\ &+ \frac{1}{n}\left(1 - n\alpha - \binom{n}{2}\beta\right) \sum_{j=1}^{n} x_j^2 \end{aligned}$$

*for all* $x_1,\ldots,x_n \in [0,1]$ *where the coefficients satisfy* $0 \leq \alpha \leq \frac{2}{n}$ *and* $-\frac{\alpha}{n-1} \leq \beta \leq \frac{1}{2\cdot\binom{n}{2}}(2-n\alpha)$ *excluding* $\beta = 0$ *when* $\alpha = \frac{1}{n}$.

# Degree 2: Symmetry and idempotent

## Proposition

*Let p be an n-ary polynomial of degree 2. The following statements are equivalent:*

i) *p is a polynomial aggregation function of degree 2 satisfying (SYM) and (ID).*

ii) *p is given by the following expression*

$$p(x_1,\dots,x_n) = \frac{1}{2}\sum_{j=1}^{n} x_j + \beta \sum_{1\leq j<j'\leq n} x_j x_{j'} - \frac{1}{n}\binom{n}{2}\beta \sum_{j=1}^{n} x_j^2$$

*for all* $x_1,\dots,x_n \in [0,1]$ *where* $\beta \in \left[-\frac{1}{n(n-1)}, \frac{1}{2\cdot\binom{n}{2}}\right] \setminus \{0\}$.

# Conclusions and future work

# Conclusions and future work

- We have introduced the family of polynomial aggregation functions.
- We have characterized all binary polynomial aggregation functions of degree one and two and we have studied under which conditions they satisfy some additional properties.
- Some results concerning $n$-ary polynomial aggregation functions have been also presented.
- To characterize of polynomial binary aggregation functions of higher degrees.
- To characterize all $n$-ary aggregation functions of low degrees, at least those satisfying some additional property.

**Thank you very much!**